Name Solutions Rec. Instr.

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Math 220

Final Exam

December 14, 2011

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		18	8		5
2		5	9		16
3		5	10		6
4		4	11		6
5		3	12		8
6		8	13		8
7		8	Total Points		100

1. (3 points each) Evaluate the following:

A. $\lim_{x\to 2} \frac{x-2}{x^2-3x+2} = \lim_{x\to 2} \frac{x-2}{(x-2)(x-1)} = \lim_{x\to 2} \frac{1}{x-1} = \frac{1}{2-1} = 1$

B. $\lim_{x\to\infty} (e^{-x}+2) = 0+2=2$

C. $\frac{d}{dx}\int_0^x \sin(t^4+1)\,dt = \sin(x^4+1)$

D. $\frac{d}{dx}\int_x^3 \cos(t)\,dt = -\frac{d}{dx}\int_3^x \cos(t)\,dt = -\cos(x)$

E. $\frac{d}{dx}(x\cdot\sec(x^2)) = 1\cdot\sec(x^2) + x\frac{d}{dx}\sec(x^2) = \sec(x^2) + x\sec(x^2)\tan(x^2)\cdot 2x$

$= \sec(x^2) + 2x^2\sec(x^2)\tan(x^2)$

F. $\lim_{x\to-\infty} \frac{4x+1}{\sqrt{4x^2+3}} = \lim_{x\to-\infty} \frac{\frac{4x+1}{x}}{\frac{\sqrt{4x^2+3}}{x}} = \lim_{x\to-\infty} \frac{4+\frac{1}{x}}{\frac{\sqrt{4x^2+3}}{-\sqrt{x^2}}}$

$= \lim_{x\to-\infty} \frac{4+\frac{1}{x}}{-\sqrt{4+\frac{3}{x^2}}} = \frac{4+0}{-\sqrt{4+0}} = -2$

2. (5 points) Find the absolute maximum and absolute minimum of $w(x) = x^3 - 3x^2 + 2$ on $[1, 3]$.

$w'(x) = 3x^2 - 6x = 3x(x-2)$

Critical number: $x = 2$ (0 is not in $[1,3]$)

$w(1) = 1^3 - 3\cdot 1^2 + 2 = 0$

$w(2) = 2^3 - 3\cdot 2^2 + 2 = -2$

$w(3) = 3^3 - 3\cdot 3^2 + 2 = 2$

Absolute Max $(3, 2)$

Absolute Min $(2, -2)$

3. (5 points) Using the limit definition of the derivative, find the derivative of $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{h(x+h)} - \frac{1}{hx} \right) = \lim_{h\to 0} \left(\frac{x - (x+h)}{hx(x+h)} \right) = \lim_{h\to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h\to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}$$

4. (4 points) Use a linear approximation to estimate $\sqrt{4.1}$.

Let $f(x) = \sqrt{x}$. $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Linearization of $f(x)$ near $x=4$: $L(x) = f(4) + f'(4)(x-4)$

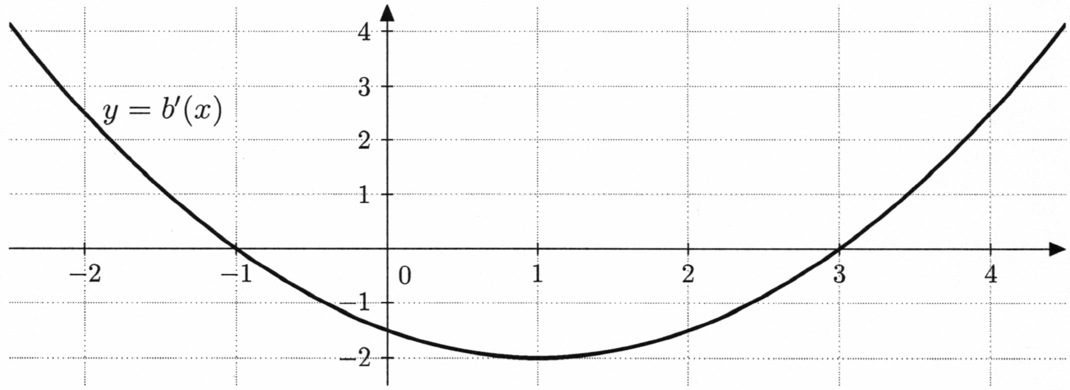
$= 2 + \frac{1}{4}(x-4)$

4.1 is close to 4

$\sqrt{4.1} = f(4.1) \approx L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025$

5. (3 points) Let $r(t)$ be the rate of growth of a puppy in pounds per week, where t denotes the number of weeks since the puppy's birth. What does $\int_5^{10} r(t)\,dt$ represent?

It represents by the Net Change Theorem the increase in the puppy's weight in pounds between ages 5 weeks and 10 weeks.



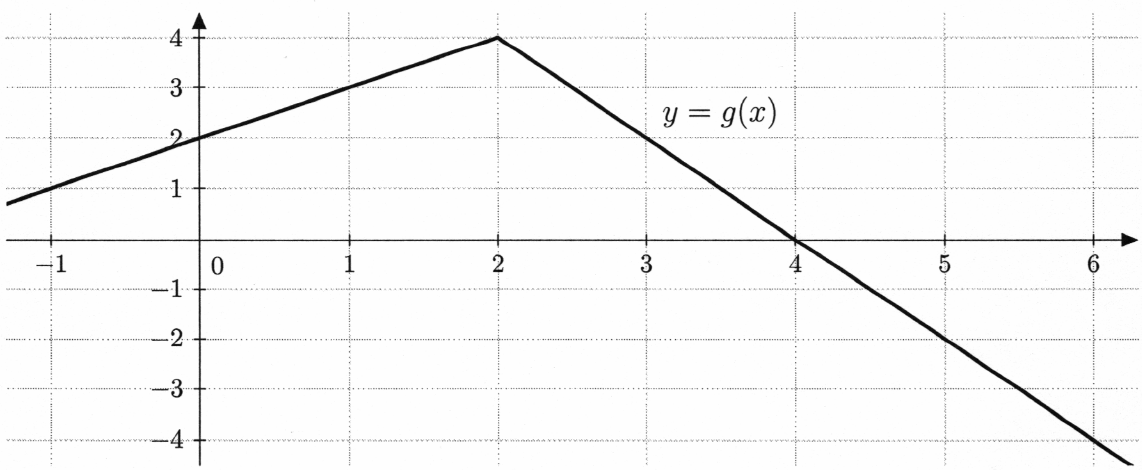
6. (2 points each) $y = b'(x)$ is plotted above. Find:

A. Interval(s) where $b(x)$ is increasing: $(-\infty,-1),(3,\infty)$ decreasing: $(-1,3)$

B. x-coordinate(s) where $b(x)$ has a local max: $x=-1$ local min: $x=3$

C. Interval(s) where $b(x)$ is concave down: $(-\infty,1)$ concave up: $(1,\infty)$

D. x-coordinate(s) where $b(x)$ has an inflection point: $x=1$



7. (2 points each) $y = g(x)$ is plotted above. Evaluate the following:

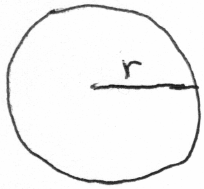
A. $\int_{-1}^{2} g(x)\,dx = \frac{1}{2}\cdot 3(1+4) = \frac{15}{2}$

B. $\int_{2}^{4} g(x)\,dx = \frac{1}{2}\cdot 2\cdot 4 = 4$

C. $\int_{4}^{6} g(x)\,dx = -\frac{1}{2}\cdot 2\cdot 4 = -4$

D. $\int_{-1}^{6} g(x)\,dx = \frac{15}{2} + 4 - 4 = \frac{15}{2}$

8. (5 points) Suppose that an oil spill from a ruptured tanker spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 ft/sec, how fast is the area of the spill increasing when the radius is 10 ft?



Know: $\frac{dr}{dt} = 2$ ft/sec

Want: $\frac{dA}{dt}$ when $r = 10$ ft.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

When $r = 10$ ft, $\frac{dA}{dt} = 2\pi \cdot 10 \cdot 2 = 40\pi$ ft^2/sec

9. (4 points each) Evaluate the following:

A. $\int_0^3 (x^2 + 2)\,dx = \frac{1}{3}x^3 + 2x \Big]_0^3 = \left(\frac{1}{3}3^3 + 2\cdot 3\right) - \left(\frac{1}{3}\cdot 0^3 + 2\cdot 0\right) = 9 + 6 = 15$

B. $\int e^x \sec^2(e^x)\,dx = \int \sec^2(u)\,du = \tan(u) + C = \tan(e^x) + C$

$u = e^x$

$du = e^x dx$

C. $\int_0^4 \frac{dx}{\sqrt{2x+1}} = \int_1^9 \frac{du}{2\sqrt{u}} = \int_1^9 \frac{1}{2} u^{-1/2}\,du = u^{1/2}\Big]_1^9 = \sqrt{9} - \sqrt{1} = 2$

$u = 2x + 1$ $\quad u(4) = 9$

$du = 2dx$ $\quad u(0) = 1$

$\frac{du}{2} = dx$

D. $\int \frac{x^2}{\sqrt{x-1}}\,dx = \int \frac{(u+1)^2}{\sqrt{u}}\,du = \int \frac{(u^2 + 2u + 1)}{\sqrt{u}}\,du = \int (u^{3/2} + 2u^{1/2} + u^{-1/2})\,du$

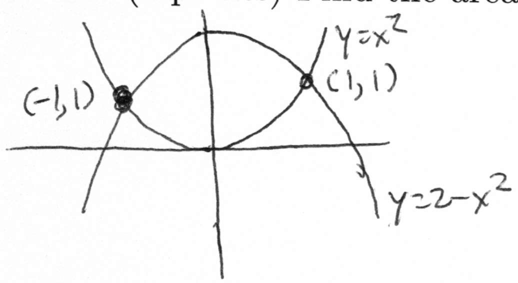
$u = x - 1$ $\quad u + 1 = x$

$du = dx$

$= \frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} + 2u^{1/2} + C$

$= \frac{2}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + 2\sqrt{x-1} + C$

10. (6 points) Find the area bounded between $y = x^2$ and $y = 2 - x^2$.

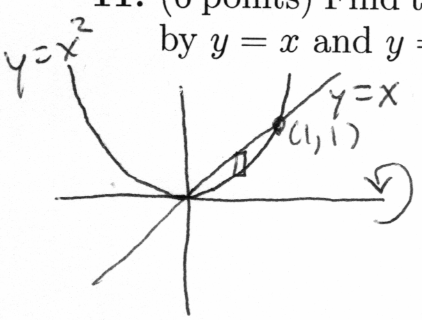


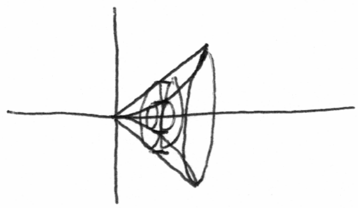
$$\text{Area} = \int_{-1}^{1} \left((2-x^2) - x^2 \right) dx = \int_{-1}^{1} (2-2x^2)\,dx$$

$$= 2\int_0^1 (2-2x^2)\,dx = 4\int_0^1 (1-x^2)\,dx = 4\left[x - \tfrac{1}{3}x^3\right]_0^1$$

$$= 4\left(\left(1-\tfrac{1}{3}\right) - (0-0) \right) = 4 \cdot \tfrac{2}{3} = \tfrac{8}{3}$$

11. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ around the x-axis.





Cross Section Areas : $A_{outer}(x) = \pi x^2$

$A_{inner}(x) = \pi (x^2)^2 = \pi x^4$

$$\text{Volume} = \underbrace{\int_0^1 \pi x^2\,dx}_{\text{Outer Volume}} - \underbrace{\int_0^1 \pi x^4\,dx}_{\text{Inner Volume}} = \int_0^1 (\pi x^2 - \pi x^4)\,dx$$

$$= \frac{\pi}{3}x^3 - \frac{\pi}{5}x^5 \Big]_0^1 = \left(\frac{\pi}{3} - \frac{\pi}{5}\right) - (0-0) = \frac{5\pi}{15} - \frac{3\pi}{15} = \frac{2\pi}{15}$$

12. (4 points each) Find $\frac{dy}{dx}$ for:

A. $x^2 - xy + y^3 = 9$

$$\frac{d}{dx}(x^2 - xy + y^3) = \frac{d}{dx} 9$$

$$2x - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(-x + 3y^2)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{3y^2 - x}$$

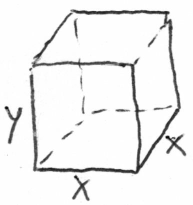
B. $y = x^{\cos(x)}$

$$\ln(y) = \ln(x^{\cos(x)}) = \cos(x)\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln(y) = \frac{d}{dx}\cos(x)\ln(x) = -\sin(x)\ln(x) + \cos(x)\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = \left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)y = \left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)x^{\cos(x)}$$

13. (8 points) If 12 ft^2 of material is available to make a box with square base and open top, find the largest possible volume for the box. (Justify why your answer is an absolute maximum.)



$$V = x^2 y$$

$$12 = x^2 + 4xy \Rightarrow y = \frac{12 - x^2}{4x}$$

$$V = x^2\left(\frac{12 - x^2}{4x}\right) = 3x - \frac{1}{4}x^3.$$ Maximize $V(x)$ for $x > 0$

$$V'(x) = 3 - \frac{3}{4}x^2$$

$$V'(x) = 0 \Rightarrow \frac{3}{4}x^2 = 3 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \Rightarrow x = 2$$

$$V''(x) = -\frac{3}{2}x < 0 \quad \text{for } x > 0$$

Hence, $V(x)$ obtains its maximum value when $x = 2$ ft, and $y = \frac{12 - 2^2}{4\cdot 2} = 1$ ft.

$$V(2) = 3\cdot 2 - \frac{1}{4}2^3 = 4 \text{ ft}^3$$