Name Solutions Rec. Instr.

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Math 220

Final Exam

May 9, 2012

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		28	6		3
2		18	7		7
3		6	8		7
4		6	9		6
5		12	10		7

Total Score:

1. (4 points each) Evaluate the following:

A. $\lim_{x\to 1} \frac{x+3}{x^2+2} = \frac{1+3}{1^2+2} = \frac{4}{3}$ $\left(\frac{x+3}{x^2+2} \text{ is continuous at } 1 \right)$

B. $\lim_{x\to -\infty} \frac{\sqrt{9x^2+6}}{4x+2} = \lim_{x\to-\infty} \frac{\frac{\sqrt{9x^2+6}}{x}}{\frac{4x+2}{x}} \underset{\substack{\uparrow \\ x<0 \Rightarrow x=-\sqrt{x^2}}}{=} \lim_{x\to-\infty} \frac{\frac{\sqrt{9x^2+6}}{-\sqrt{x^2}}}{4+\frac{2}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{9+\frac{6}{x^2}}}{4+\frac{2}{x}}$

$= \frac{-\sqrt{9+0}}{4+0} = -\frac{3}{4}$

C. $\frac{d}{dx} \int_0^x \sin(t)\cos(t^2)\, dt = \sin(x)\cos(x^2)$

D. $\int_0^{\pi/2} \cos(t)\, dt = \sin(t)\Big]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$

E. $\frac{d}{dx}\left(\frac{\ln(x)}{x+3}\right) = \frac{\left[\frac{d}{dx}\ln(x)\right](x+3) - \ln(x)\left[\frac{d}{dx}(x+3)\right]}{(x+3)^2} = \frac{\frac{x+3}{x} - \ln(x)}{(x+3)^2}$

F. $\frac{d}{dx}\left(e^x \tan(x^2)\right) = \left[\frac{d}{dx} e^x\right]\tan(x^2) + e^x\left[\frac{d}{dx}\tan(x^2)\right]$

$= e^x \tan(x^2) + e^x \cdot \sec^2(x^2) \cdot \left[\frac{d}{dx} x^2\right]$

$= e^x \tan(x^2) + e^x \cdot \sec^2(x^2) \cdot 2x$

G. $\int \frac{dx}{1+x^2} = \arctan(x) + C$

2. (6 points each) Evaluate the following:

A. $\int e^{\sin(x)} \cos(x)\, dx =$ $\int e^u du = e^u + c = e^{\sin(x)} + C$

$u = \sin(x)$

$du = \cos(x)dx$

B. $\int_0^1 x\sqrt{x^2+1}\, dx =$ $\int_1^2 \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big]_1^2 = \frac{1}{3} u^{3/2} \Big]_1^2$

$u = g(x) = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$g(1) = 2$

$g(0) = 1$

$= \frac{1}{3} \cdot 2^{3/2} - \frac{1}{3} \cdot 1^{3/2}$

$= \frac{1}{3} \cdot 2^{3/2} - \frac{1}{3}$

C. $\frac{d}{dx} \int_0^{x^2} e^t \cos(t)\, dt$ $= f'(g(x)) \cdot g'(x) = e^{x^2} \cdot \cos(x^2) \cdot 2x$

$\int_0^{x^2} e^t \cos(t) dt = f(g(x))$, where $f(y) = \int_0^y e^t \cos(t) dt$ and $g(x) = x^2$.

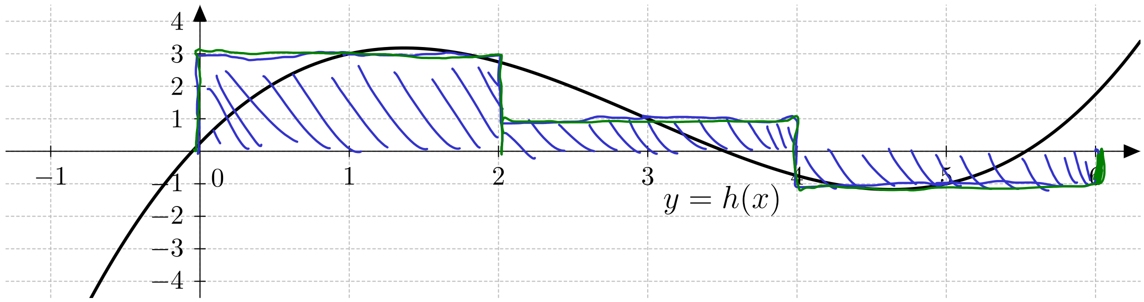
$f'(y) = e^y \cos(y)$, and $g'(x) = 2x$.

3. (6 points) Using the **limit definition of the derivative**, find the derivative of $f(x) = 3x^2$.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$

$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} = \lim_{h \to 0} \frac{6xh + 3h^3}{h}$

$= \lim_{h \to 0} (6x + 3h) = 6x$



4. (6 points) $y = h(x)$ is plotted above. Estimate $\int_0^6 h(x)\,dx$ by using $n = 3$ subintervals, taking the sampling points to be midpoints. Also, illustrate the rectangles on the graph above.

$$\Delta x = \frac{6-0}{3} = 2$$

$$\int_0^6 h(x)\,dx \approx h(1)\Delta x + h(3)\Delta x + h(5)\Delta x$$

$$= 3 \cdot 2 + 1 \cdot 2 + (-1)\cdot 2$$

$$= 6$$

5. (6 points each) Find $\dfrac{dy}{dx}$ for:

A. $x^3 + y^3 = 5 - xy$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(5-xy)$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = -y - x\frac{dy}{dx}$$

$$3x^2 + y = -3y^2 \cdot \frac{dy}{dx} - x\frac{dy}{dx} = -(3y^2+x)\frac{dy}{dx}$$

$$-\frac{3x^2+y}{3y^2+x} = \frac{dy}{dx}$$

B. $y = x^{3x}$

$$\ln(y) = \ln(x^{3x}) = 3x\ln(x)$$

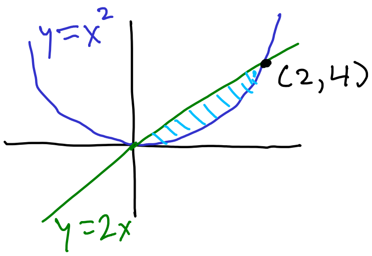
$$\frac{1}{y}\cdot\frac{dy}{dx} = 3\ln(x) + 3x\cdot\frac{1}{x} = 3\ln(x) + 3$$

$$\frac{dy}{dx} = (3\ln(x)+3)\cdot y = (3\ln(x)+3)\cdot x^{3x}$$

6. (3 points) Let $w(t)$ be the rate that water flows out of a storage tank in gallons per minute at time t minutes after the tank ruptures. What does $\int_0^{10} w(t)\,dt$ represent?

By the Net Change Theorem, this is the number of gallons of water that flow out of the tank in the first 10 minutes after the tank ruptures.

7. (7 points) Find the area bounded between $y = x^2$ and $y = 2x$.



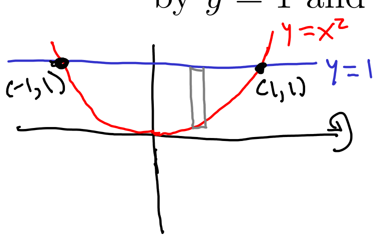
$$2x = x^2 \Rightarrow 0 = x^2 - 2x = x(x-2) \Rightarrow x = 0 \text{ or } 2$$

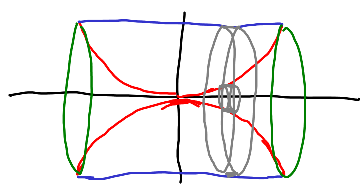
$$\text{AREA} = \int_0^2 |x^2 - 2x|\,dx = \int_0^2 (2x - x^2)\,dx$$

$$= x^2 - \frac{1}{3}x^3\Big]_0^2 = \left(2^2 - \frac{1}{3}\cdot 2^3\right) - \left(0^2 - \frac{1}{3}0^3\right)$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

8. (7 points) Find the volume of the solid obtained by rotating the region bounded by $y = 1$ and $y = x^2$ around the x-axis.

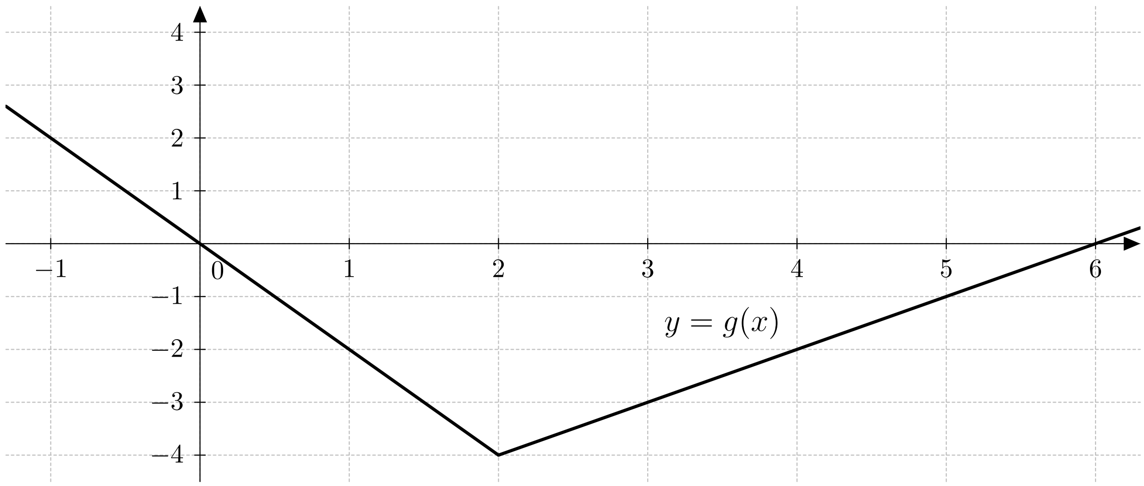




The cross-sectional areas obtained by rotating the outer curve $y=1$ and the inner curve $y=x^2$ around the x-axis are $A_{outer}(x) = \pi(1)^2 = \pi$ and $A_{inner}(x) = \pi(x^2)^2 = \pi x^4$.

$$\text{Volume} = \underbrace{\int_{-1}^{1} \pi\,dx}_{\text{volume from rotating } y=1 \text{ around the } x\text{-axis from } x=-1 \text{ to } x=1} - \underbrace{\int_{-1}^{1} \pi x^4\,dx}_{\text{volume from rotating } y=x^2 \text{ around the } x\text{-axis from } x=-1 \text{ to } x=1} = 2\pi - \left[\frac{\pi}{5}x^5\right]_{-1}^{1} = 2\pi - \left(\frac{\pi}{5}\cdot 1^5 - \frac{\pi}{5}(-1)^5\right)$$

$$= 2\pi - \frac{2\pi}{5} = \frac{8\pi}{5}$$

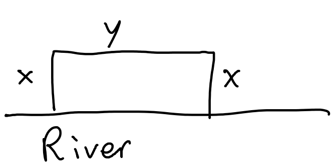


9. (3 points each) $y = g(x)$ is plotted above. Evaluate the following definite integrals. (No work needs to be shown.)

A. $\int_{-1}^{0} g(x)\,dx = \frac{1}{2} \cdot 1 \cdot 2 = 1$

B. $\int_{0}^{6} g(x)\,dx = -\frac{1}{2} \cdot 6 \cdot 4 = -12$

10. (7 points) A farmer has 20 feet of fencing and wants to fence off a rectangular area that borders a straight river. The farmer needs no fencing along the river. What dimensions will maximize the fenced-in area? (Make sure to justify why your answer corresponds to the absolute maximum.)



The area of the rectangle is $A = xy$.

We know that $2x + y = 20$ so $y = 20 - 2x$.

Thus, $A(x) = x(20-2x) = 20x - 2x^2$.

$A'(x) = 20 - 4x$ is never undefined, and

$A'(x) = 0 \Rightarrow 0 = 20 - 4x \Rightarrow 4x = 20 \Rightarrow x = 5$.

Option 1	Option 2	Option 3
we need to maximize $A(x)$ on the closed interval $[0, 10]$. $A(0) = 0$, $A(5) = 50$, $A(10) = 0$	$A''(x) = -4 < 0$ for all x.	$A'(x) > 0$ for $x < 5$ $A'(x) < 0$ for $x > 5$

Hence, the area is maximized when $x = 5$ ft and $y = 20 - 2 \cdot 5 = 10$ ft.