Name Solutions Signature ______________

Math 220 – Final Exam – December 17, 2014

1. (6 points) Let $g(x) = x^2 + 1$. **Using the limit definition of the derivative,** find $g'(3)$.

$$g'(3) = \lim_{h\to 0} \frac{g(3+h)-g(3)}{h} = \lim_{h\to 0} \frac{((3+h)^2+1)-(3^2+1)}{h}$$

$$= \lim_{h\to 0} \frac{(9+6h+h^2+1)-(9+1)}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h)$$

$$= 6+0 = 6$$

2. (6 points) Using a linearization for $f(x) = \sqrt{x}$ at $x = 1$, estimate $\sqrt{1.02}$.

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. The linearization of $f(x)$ at $x=1$ is

$L(x) = f(1) + f'(1)(x-1) = \sqrt{1} + \frac{1}{2\sqrt{1}}(x-1) = 1 + \frac{1}{2}(x-1)$.

Because 1.02 is close to 1,

$$\sqrt{1.02} = f(1.02) \approx L(1.02) = 1 + \frac{1}{2}(1.02-1) = 1 + \frac{1}{2}(.02) = 1.01$$

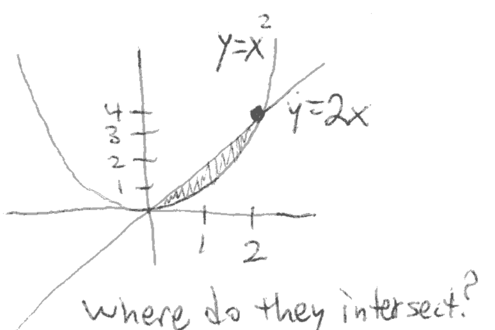
3. (3 points) At time t minutes, Alice has a velocity of $v_A(t)$ ft/min, and Fred has a velocity of $v_F(t)$ ft/min. Describe the meaning of the integral

$$\int_2^5 (v_A(t) - v_F(t))\, dt.$$

$$\int_2^5 (v_A(t) - v_F(t))\,dt = \int_2^5 v_A(t)\,dt - \int_2^5 v_F(t)\,dt$$

This is how much farther in feet Alice runs than Fred from 2 minutes to 5 minutes.

4. (6 points) Find the area bounded between $y = 2x$ and $y = x^2$.



$$\text{Area} = \int_0^2 (2x - x^2)\,dx = \left(x^2 - \tfrac{1}{3}x^3\right)\Big|_0^2$$

$$= \left(2^2 - \tfrac{1}{3}(2)^3\right) - \left(0^2 - \tfrac{1}{3}\cdot(0)^3\right)$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

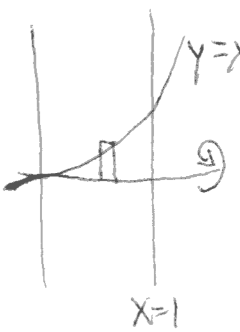
where do they intersect?

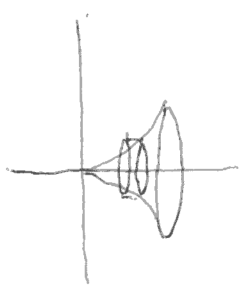
$2x = x^2$

$0 = x^2 - 2x = x(x-2)$

$x = 0$ or $x = 2$

5. (6 points) Find the volume of the solid formed by rotating the region bounded by $y = 0$, $x = 1$, and $y = x^3$ around the x-axis.





$$\text{Volume} = \int_0^1 \pi(x^3)^2\,dx = \int_0^1 \pi x^6\,dx$$

$$= \frac{\pi}{7}x^7\Big|_0^1 = \frac{\pi}{7}(1)^7 - \frac{\pi}{7}(0)^7$$

$$= \frac{\pi}{7}$$

6. (6 points) Find $m(x)$ provided that $m''(x) = e^x$, $m'(0) = 2$, and $m(0) = 5$.

$\int e^x\,dx = e^x + C$ so $m'(x) = e^x + C$ for some constant C.

$2 = m'(0) = e^0 + C = 1 + C$ so $C = 1$. Hence, $m'(x) = e^x + 1$.

$\int (e^x + 1)\,dx = e^x + x + D$ so $m(x) = e^x + x + D$ for some constant D.

$5 = m(0) = e^0 + 0 + D = 1 + D$ so $D = 4$. Therefore, $m(x) = e^x + x + 4$.

7. (4 points each) Compute the following:

A. $\frac{d}{dx}\int_3^x t^5\cos(t^3+42t)\,dt$ $= x^5 \cdot \cos(x^3+42x)$

B. $\frac{d}{dx}\left(\frac{e^{x^2}}{2x^5+x}\right)$ $= \dfrac{\left[\frac{d}{dx}e^{x^2}\right]\cdot(2x^5+x) - e^{x^2}\cdot\left[\frac{d}{dx}(2x^5+x)\right]}{(2x^5+x)^2}$

$= \dfrac{e^{x^2}\cdot 2x\cdot(2x^5+x) - e^{x^2}\cdot(10x^4+1)}{(2x^5+x)^2}$

C. $\frac{d}{dx}(\sin(x)\cdot\arctan(x))$ $= \cos(x)\cdot\arctan(x) + \sin(x)\cdot\frac{1}{1+x^2}$

D. $\lim_{x\to 2}\frac{x^2-4}{x-2}$ $= \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2=4$

E. $\lim_{x\to\infty}\frac{7x^5+3x^2+3}{6x^5+4x^4+2x}$ $= \lim_{x\to\infty}\dfrac{\frac{7x^5+3x^2+3}{x^5}}{\frac{6x^5+4x^4+2x}{x^5}} = \lim_{x\to\infty}\dfrac{7+\frac{3}{x^3}+\frac{3}{x^5}}{6+\frac{4}{x}+\frac{2}{x^4}}$

$= \frac{7+0+0}{6+0+0} = \frac{7}{6}$

8. (6 points each) Compute the following:

A. $\dfrac{dy}{dx}$ if $y^4 + xy = x^3 - x + 2$

$$\frac{d}{dx}[y^4+xy] = \frac{d}{dx}[x^3-x+2]$$

$$4y^3\frac{dy}{dx} + 1oy + x\cdot\frac{dy}{dx} = 3x^2-1$$

$$4y^3\frac{dy}{dx} + x\frac{dy}{dx} = 3x^2-1-y$$

$$(4y^3+x)\frac{dy}{dx} = 3x^2-1-y$$

$$\frac{dy}{dx} = \frac{3x^2-1-y}{4y^3+x}$$

B. $\displaystyle\int x \cdot \sec^2(x^2)\, dx$ $= \displaystyle\int \sec^2(u)\,\frac{dy}{2} = \frac{\tan(u)}{2} + C = \frac{\tan(x^2)}{2} + C$

$u = x^2$

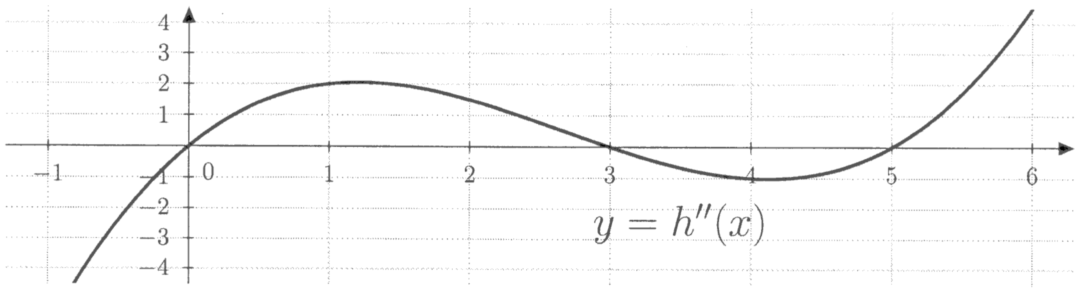
$du = 2x\,dx$

$\frac{dy}{2} = x\,dx$

C. $\displaystyle\int_0^{\pi/2} \sqrt{\sin(t)}\cos(t)\, dt$ $= \displaystyle\int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{2}{3}$

$u = \sin(t)$ $\quad u(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

$du = \cos(t)\,dt$ $\quad u(0) = \sin(0) = 0$



9. (2 points each) Above is a graph of $y = h''(x)$. Find:

A. Interval(s) where $h(x)$ is concave up: $(0,3)$ and $(5,\infty)$

B. Interval(s) where $h(x)$ is concave down: $(-\infty,0)$ and $(3,5)$

C. x-coordinate(s) where $h(x)$ has an inflection point: $x=0,3,5$

10. (3 points) Let $r(t)$ denote the rate in gallons per hour at which water is flowing into a tank t hours after noon. Describe the meaning of $\int_2^4 r(t)\,dt$.

This represents the amount of water in gallons that flows into the tank between 2:00 PM and 4:00 PM.

11. (6 points) Find the absolute maximum and minimum of $w(x) = x^3 + 3x^2 - 9x + 1$ on $[0, 2]$.

$w'(x) = 3x^2 + 6x - 9 = 3(x^2+2x-3) = 3(x+3)(x-1)$ is defined everywhere.

$w'(x) = 0$ when $x=1$ or $x=-3$. The only critical point in $[0,2]$ occurs when $x=2$.

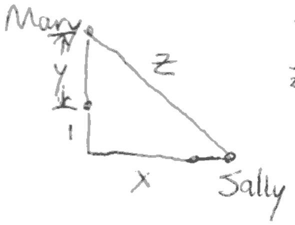
$w(2) = 2^3 + 3\cdot(2)^2 - 9(2) + 1 = 3$

$w(1) = 1^3 + 3\cdot(1)^2 - 9\cdot(1) + 1 = -4$

$w(0) = 0^3 + 3\cdot(0)^2 - 9\cdot(0) + 1 = 1$

On $[0,2]$, $w(x)$ has an absolute max at $(2,3)$ and an absolute minimum at $(1,-4)$.

12. (7 points) At noon, Mary is a mile north of Sally. Mary is hiking north at a rate of 1 mile per hour, and Sally is walking east at a rate of 2 miles per hour. How fast is the distance between Mary and Sally changing at 2:00 PM?



x = distance Sally has walked t hours after noon

y = distance Mary has hiked t hours after noon

z = distance between Mary and Sally t hours after noon

want: $\frac{dz}{dt}$ when $t=2$ hr. Know: $\frac{dx}{dt} = 2$ mi/hr, $\frac{dy}{dt} = 1$ mi/hr.

$$z^2 = x^2 + (y+1)^2 \quad \text{so} \quad \frac{d}{dt} z^2 = \frac{d}{dt}\left[x^2 + (y+1)^2\right]$$

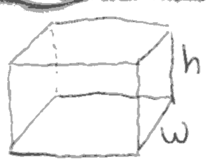
$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2(y+1)\frac{dy}{dt} \quad \text{So} \quad \frac{dz}{dt} = \frac{2x\frac{dx}{dt} + 2(y+1)\frac{dy}{dt}}{2z} = \frac{x\frac{dx}{dt} + (y+1)\frac{dy}{dt}}{z}$$

when $t = 2$ hr, $x = 2\cdot 2 = 4$ mi, $y = 1\cdot 2 = 2$ mi, and

$z = \sqrt{4^2 + (2+1)^2} = \sqrt{25} = 5$ mi. Thus, when $t = 2$ hr,

$$\frac{dz}{dt} = \frac{4\cdot 2 + (2+1)\cdot 1}{5} = \frac{11}{5} \text{ mi/hr.}$$

13. (7 points) 6 ft^2 of material is available to make a rectangular box with an open top. The length of its base must be twice the width. Find the dimensions that maximize the volume of the box. (Justify why your answer is an absolute maximum.)



$l = 2w$

Maximize $V = l\cdot w\cdot h = 2w^2\cdot h$

$$6 = w(2w) + 2\cdot h\cdot w + 2\cdot h\cdot(2w) = 2w^2 + 6hw$$

$$6 - 2w^2 = 6hw \quad \text{so} \quad h = \frac{6-2w^2}{6w} = \frac{1}{w} - \frac{1}{3}w.$$

Maximize $V(w) = 2w^2\left(\frac{1}{w} - \frac{1}{3}w\right) = 2w - \frac{2}{3}w^3$ on $(0,\infty)$

$V'(w) = 2 - 2w^2$ is defined everywhere.

$V'(w) = 0$ when $2w^2 = 2$, $w^2 = 1$, $w = \pm 1$.

The only critical point of $V(x)$ on $(0,\infty)$ occurs when $w = 1$ ft. $V''(x) = -4w < 0$ on $(0,\infty)$ so the volume has a max when $w = 1$ ft, $l = 2w = 2$ ft,

$h = \frac{6-2w^2}{6w} = \frac{4}{6} = \frac{2}{3}$ ft, and $V = 1\cdot 2\cdot \frac{2}{3} = \frac{4}{3}$ ft^3.