Name Solutions Signature

Math 220 – Final Exam (Version A) – May 14, 2014

1. (7 points each) Find the following:

A. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^2+6}}{3x+5} = \lim\limits_{x\to-\infty} \dfrac{\frac{\sqrt{4x^2+6}}{x}}{\frac{3x+5}{x}} = \lim\limits_{x\to-\infty} \dfrac{\frac{\sqrt{4x^2+6}}{-\sqrt{x^2}}}{3+\frac{5}{x}}$

$= \lim\limits_{x\to-\infty} \dfrac{-\sqrt{4+\frac{6}{x^2}}}{3+\frac{5}{x}} = \dfrac{-\sqrt{4+0}}{3+0} = -\dfrac{2}{3}$

B. $\int_0^2 x^2 e^{x^3}\,dx = \int_0^8 e^u \frac{du}{3} = \frac{e^u}{3}\Big|_0^8 = \frac{e^8}{3} - \frac{e^0}{3} = \frac{e^8}{3} - \frac{1}{3}$

$u = x^3$ $\quad u(2) = 8$

$du = 3x^2\,dx$ $\quad u(0) = 0$

$\frac{du}{3} = x^2\,dx$

C. $\dfrac{dy}{dx}$ if $x^2 + 4y^2 = 3$

$\frac{d}{dx}(x^2+4y^2) = \frac{d}{dx}3$

$2x + 8y\frac{dy}{dx} = 0$

$8y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{8y} = -\frac{x}{4y}$

2. (7 points) Let $f(x) = \frac{1}{x}$. Using the limit definition of the derivative, find $f'(2)$.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{2(2+h)} - \frac{2+h}{2(2+h)}}{h} = \lim_{h \to 0} \frac{\frac{-h}{2(2+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{2(2+h)h} = \lim_{h \to 0} \frac{-1}{2(2+h)} = \frac{-1}{2(2+0)} = -\frac{1}{4}$$

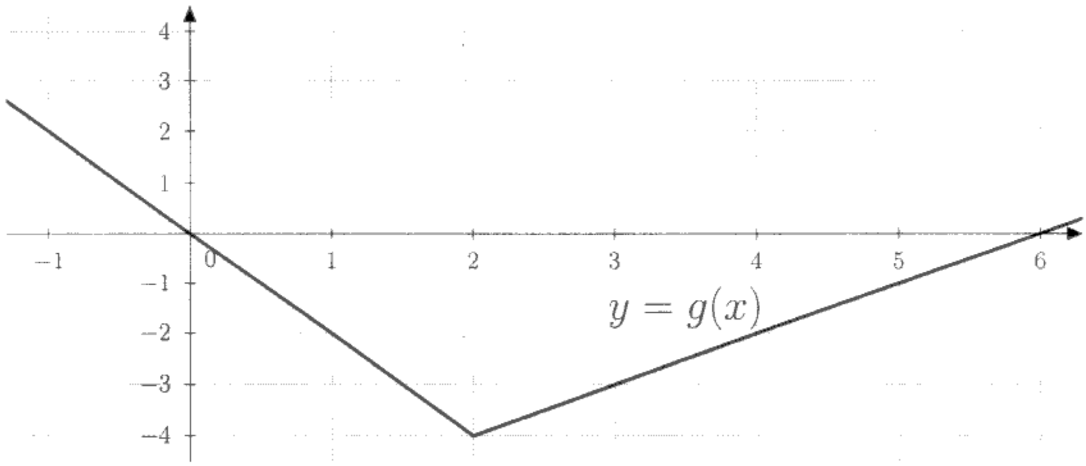
3. (5 points each) Find the following:

A. $\frac{d}{dx} \cos(x^2 + 3x) = -\sin(x^2+3x)\left[\frac{d}{dx}(x^2+3x)\right]$

$$= -\sin(x^2+3x) \cdot (2x+3)$$

B. $\frac{d}{dx}\left(\frac{2^x}{x^4 + x^2}\right) = \frac{2^x \cdot \ln(2)(x^4+x^2) - 2^x \cdot (4x^3+2x)}{(x^4+x^2)^2}$

C. $\int_0^3 t^2\, dt = \frac{1}{3}t^3 \Big|_0^3 = \frac{1}{3} \cdot 3^3 - \frac{1}{3} \cdot 0^3 = 9$

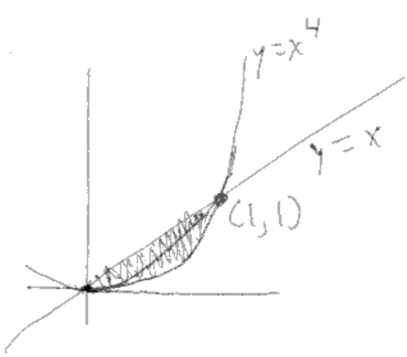


4. (3 points each) $y = g(x)$ is plotted above. Evaluate the following definite integrals. (No work needs to be shown.)

A. $\int_{-1}^{2} g(x)\,dx = \frac{1}{2}\cdot 1\cdot 2 - \frac{1}{2}\cdot 2\cdot 4 = -3$

B. $\int_{2}^{6} g(x)\,dx = -\frac{1}{2}\cdot 4\cdot 4 = -8$

5. (9 points) Find the area bounded between $y = x^4$ and $y = x$.



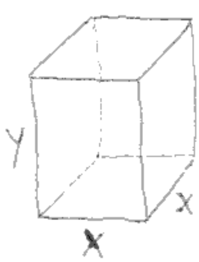
$$\text{Area} = \int_0^1 (x - x^4)\,dx$$
$$= \left(\frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Big|_0^1$$
$$= \left(\frac{1}{2}(1)^2 - \frac{1}{5}(1)^5\right) - \left(\frac{1}{2}(0)^2 - \frac{1}{5}(0)^5\right)$$
$$= \frac{1}{2} - \frac{1}{5} = \frac{5}{10} - \frac{2}{10} = \frac{3}{10}$$

6. (6 points) Find the linearization of $k(x) = e^x$ at $x = 0$.

$k'(x) = e^x$. The linearization of $k(x)$ at $x=0$ is

$L(x) = k(0) + k'(0)(x-0) = e^0 + e^0(x-0) = 1+x$.

7. (10 points) A rectangular open-topped box is to have a square base and volume 8 ft^3. If material for the base costs \$2 per ft^2 and material for the sides costs \$1 per ft^2, what dimensions minimize the cost of the box? (Justify why your answer is an absolute minimum.)



Minimize Cost: $C = 2x^2 + 4 \cdot 1 \cdot xy = 2x^2 + 4xy$

Volume: $x^2 y = 8$ so $y = \frac{8}{x^2}$

Minimize $C(x) = 2x^2 + 4xy = 2x^2 + 4x\left(\frac{8}{x^2}\right) = 2x^2 + \frac{32}{x}$

on $(0, \infty)$.

$C'(x) = 4x - \frac{32}{x^2}$. $C'(x) = 0 \Leftrightarrow 4x - \frac{32}{x^2} = 0 \Leftrightarrow 4x = \frac{32}{x^2}$

$\Leftrightarrow 4x^3 = 32 \Leftrightarrow x^3 = 8 \Leftrightarrow x = 2$.

$x=2$ is the only critical point on $(0,\infty)$, and

$C''(x) = 4 + \frac{64}{x^3} > 0$ on $(0,\infty)$. Hence, the cost

is minimized by choosing $x = 2$ ft and $y = \frac{8}{2^2} = 2$ ft.

Dimensions: 2 ft by 2 ft by 2 ft

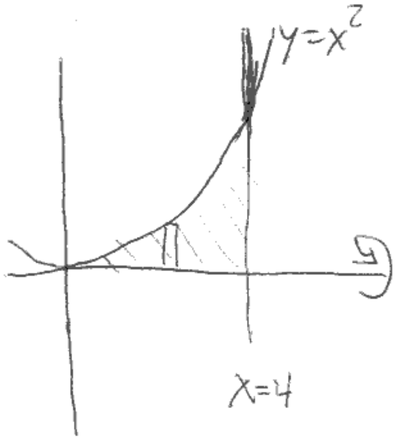
8. (6 points) Find $\frac{d}{dx}\int_0^{x^2} \cos(t^3+1)\,dt$.

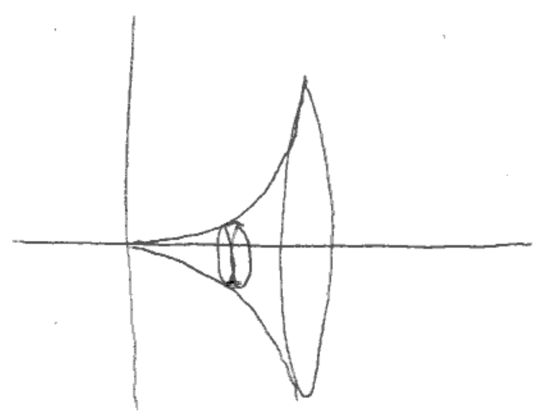
Let $f(x) = \int_0^x \cos(t^3+1)\,dt$. $f'(x) = \cos(x^3+1)$.

$$\frac{d}{dx}\int_0^{x^2}\cos(t^3+1)\,dt = \frac{d}{dx} f(x^2) = f'(x^2)\cdot\left[\frac{d}{dx}x^2\right]$$

$$= \cos((x^2)^3+1)\cdot 2x = \cos(x^6+1)\cdot 2x$$

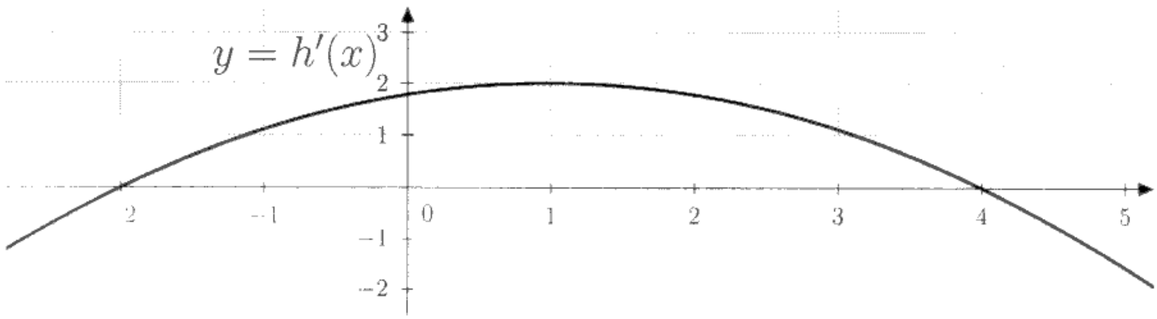
9. (9 points) Find the volume of the solid obtained by rotating the region bounded by $y = 0$, $x = 4$, and $y = x^2$ around the x-axis.





$$\text{Volume} = \int_0^4 \pi (x^2)^2\,dx = \int_0^4 \pi x^4\,dx = \frac{\pi}{5}x^5\Big|_0^4$$

$$= \frac{\pi}{5}\cdot 4^5 - \frac{\pi}{5}\cdot 0^5 = \frac{\pi\cdot 4^5}{5}$$



10. (1 point each) $y = h'(x)$ is plotted above. Find the following:

A. Interval(s) where $h(x)$ is increasing: $(-2, 4)$

B. Interval(s) where $h(x)$ is decreasing: $(-\infty, -2)$ and $(4, \infty)$

C. x-coordinate(s) where $h(x)$ has a local max: $x = 4$

D. x-coordinate(s) where $h(x)$ has a local min: $x = -2$

E. Interval(s) where $h(x)$ is concave up: $(-\infty, 1)$

F. Interval(s) where $h(x)$ is concave down: $(1, \infty)$

G. x-coordinate(s) where $h(x)$ has an inflection point: $x = 1$

11. (4 points) Let $w(t)$ be the rate that oil flows out of a storage tank in gallons per minute at time t minutes after the tank ruptures. What does $\int_0^{100} w(t)\, dt$ represent?

This integral represents the number of gallons that have leaked from the tank during the first 100 minutes after the rupture.