Math 220 Calculus I
Professor John Maginnis

Midterm Exam 1
September 24, 2015

Your name: Solutions

Rec. Instr.: ____________ Rec. Time: ____________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4	5
Points	/4	/4	/4	/4	/4
Problem	6	7	8	9	10
Points	/4	/4	/5	/5	/5
Problem	11	12	13		Total
Points	/5	/4	/8		/60

1. Find an equation of the tangent line to the curve $y = 2x^4 - 3x^3 + 5x - 7$ at the point where $x = 1$.

$$\frac{dy}{dx} = 8x^3 - 9x^2 + 5$$

$$m = y'(1) = 8 - 9 + 5 = 4$$

$$y(1) = 2 - 3 + 5 - 7 = -3$$

$$\boxed{y + 3 = 4(x - 1)} \quad \text{or} \quad \boxed{y = 4x - 7}$$

Evaluate the following limits.

2.

$$\lim_{x\to -2}\left(\frac{3x^2+5x-2}{x^2+5x+6}\right) = \frac{12-10-2}{4-10+6} = \frac{0}{0} \quad \underline{\text{not determined}}$$

$$\lim_{x\to -2} \frac{(x+2)(3x-1)}{(x+2)(x+3)} = \lim_{x\to -2} \frac{3x-1}{x+3} = \frac{-6-1}{-2+3} = \boxed{-7}$$

3.

$$\lim_{\theta\to 0}\left(\sin(4\theta)\cot(2\theta)\right) = 0 \times \infty \quad \underline{\text{not determined}}$$

$$\lim_{\theta\to 0} \sin(4\theta)\cdot\frac{\cos(2\theta)}{\sin(2\theta)}\cdot\frac{4\theta}{4\theta}\cdot\frac{2\theta}{2\theta}$$

$$\lim_{\theta\to 0} \frac{\sin(4\theta)}{4\theta}\cdot\frac{2\theta}{\sin(2\theta)}\cdot\cos(2\theta)\cdot\frac{4\theta}{2\theta} = 1\cdot 1\cdot \underset{\cos(0)}{\underset{\parallel}{1}}\cdot 2 = \boxed{2}$$

4.

$$\lim_{x\to\infty}\left(\frac{6x^2-5x}{2x^2+3}\right) = \frac{\infty}{\infty} \quad \underline{\text{not determined}}$$

$$\lim_{x\to\infty} \frac{6x^2-5x}{2x^2+3}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty}\frac{6-\frac{5}{x}}{2+\frac{3}{x^2}} = \frac{6-0}{2+0} = \boxed{3}$$

($3 = \frac{6}{2}$ = ratio of leading coefficients, when the degree of the numerator equals the degree of the denominator)

5. Evaluate the right hand limit.

$$\lim_{x\to 3+} \frac{2x-4}{9-x^2} = \frac{6-4}{9-9} = \frac{2}{0} = \pm\infty$$

Since $x > 3$, $x-3>0$, and $3-x<0$.

$$\lim_{x\to 3^+} \frac{2x-4}{(3-x)(3+x)} = \frac{2}{(\text{a small negative number})(6)} = \boxed{-\infty}$$

(or use $x>3$, $x^2>9$, $9-x^2<0$)

6. Find the derivative.

$$y = \frac{x^2-x}{x^3+1}$$

$$\frac{dy}{dx} = \frac{(x^3+1)(2x-1)-(x^2-x)(3x^2)}{(x^3+1)^2}$$

$$= \frac{2x^4-x^3+2x-1-(3x^4-3x^3)}{(x^3+1)^2} = \boxed{\frac{-x^4+2x^3+2x-1}{(x^3+1)^2}}$$

7. Find the second derivative y''.

$$y = x^5e^x$$

$$y' = \frac{dy}{dx} = 5x^4e^x + x^5e^x$$

$$y'' = \frac{d^2y}{dx^2} = 20x^3e^x + 5x^4e^x + 5x^4e^x + x^5e^x$$

$$= \boxed{20x^3e^x + 10x^4e^x + x^5e^x}$$

$$= x^3e^x(x^2+10x+20)$$

8. Find the derivative $\frac{dz}{dx}$. Do not simplify.

$$z = \frac{x^2 e^x}{x^3 - 2}$$

$$\frac{dz}{dx} = \boxed{\frac{(x^3-2)(2xe^x + x^2 e^x) - x^2 e^x (3x^2)}{(x^3-2)^2}}$$

$$= \frac{e^x(x^5 - x^4 - 2x^2 - 4x)}{(x^3-2)^2}$$

9. Let $f(x) = \begin{cases} \frac{x+3}{x^2+1} & \text{if } \ x < 1 \\ 3 & \text{if } \ x = 1 \\ 3 - x^2 & \text{if } \ x > 1 \end{cases}$

Determine whether $y = f(x)$ is continuous at $x = 1$ by computing limits.

$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \left(\frac{x+3}{x^2+1}\right) = \frac{4}{2} = 2 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (3 - x^2) = 3 - 1 = 2 \end{aligned}\right\} \text{Equal, so } \lim_{x\to 1} f(x) = 2.$$

But $f(1) = 3$. So $\lim_{x\to 1} f(x) = 2 \neq 3 = f(1)$;

$f(x)$ is <u>not continuous</u> at $x = 1$.

(There is a <u>removable discontinuity</u>)

10. Compute the limit.

$$\lim_{x\to 1}\left(\frac{\sqrt{x+3}-2}{x^2-1}\right)\cdot\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) =$$

$$\lim_{x\to 1}\frac{(x+3)-4}{(x^2-1)(\sqrt{x+3}+2)} = \lim_{x\to 1}\frac{x-1}{(x-1)(x+1)(\sqrt{x+3}+2)}$$

$$= \lim_{x\to 1}\frac{1}{(x+1)(\sqrt{x+3}+2)} = \frac{1}{(2)(2+2)} = \boxed{\frac{1}{8}}$$

11. Use the definition of the derivative as a limit to find $f'(2)$ for the function $f(x) = 4x^2 - 3x + 6$.

Note $f(2) = 16 - 6 + 6 = 16$. Then

$$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{4x^2-3x+6-16}{x-2} =$$

$$\lim_{x\to 2}\frac{4x^2-3x-10}{x-2} = \lim_{x\to 2}\frac{(x-2)(4x+5)}{x-2} = \lim_{x\to 2}(4x+5) = \boxed{13}$$

Alternatively $f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{4(2+h)^2-3(2+h)+6-16}{h}$

$$= \lim_{h\to 0}\frac{4(4+4h+h^2)-6-3h+6-16}{h} = \lim_{h\to 0}\frac{16h+4h^2-3h}{h} = \lim_{h\to 0}(13+4h) = \boxed{13}$$

12. Let $y = 2x^3 - 3x^2 - 2x + 1$. Explain why this function has (at least) three x-intercepts in the interval $-1 \le x \le 2$.

x	y
-1	$-2 < 0$
0	$1 > 0$
1	$-2 < 0$
2	$1 > 0$

By the Intermediate Value Theorem, each time the continuous function changes sign, there must be a zero or x-intercept. So each of the intervals $[-1, 0]$, $[0, 1]$, and $[1, 2]$ contains at least one x-intercept.

13. A ball is thrown vertically from the roof of a building, and the height of the ball above the ground t seconds later satisfies the equation $y = -16t^2 + 48t + 64$.

(a) Find the maximum height of the ball.

$$v = \frac{dy}{dt} = -32t + 48 = 0, \quad t = \frac{48}{32} = \frac{3}{2} \text{ seconds.}$$

$$y = -16\left(\frac{9}{4}\right) + 48\left(\frac{3}{2}\right) + 64 = -36 + 72 + 64$$

$$= \boxed{100 \text{ feet}}$$

(b) Find the velocity of the ball at the instant it hits the ground.

$$y = 0, \quad 0 = -16t^2 + 48t + 64 = -16(t^2 - 3t - 4)$$

$$0 = -16(t-4)(t+1)$$

$$\boxed{t = 4} \text{ or } t = -1 \text{ (not in domain)}$$

$$v = \frac{dy}{dt} = y'(4) = -32(4) + 48 = \boxed{-80 \frac{\text{feet}}{\text{sec.}}}$$