Math 220 Calculus I
Professor John Maginnis

Midterm Exam 2
October 22, 2015

Your name: Solutions

Rec. Instr.: ____________ Rec. Time: ____________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

| Problem | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Points | /8 | /5 | /4 | /4 | /5 |
| Problem | 6 | 7 | 8 | 9 | 10 |
| Points | /5 | /4 | /8 | /9 | /8 |

1. Approximate the fifth root of thirty three using the following two methods.

(a) Find an equation of the tangent line to the curve $y = \sqrt[5]{x}$ at $x = 32$, and use this to approximate $\sqrt[5]{33}$.

$y = \sqrt[5]{x} = x^{1/5}$ , $\sqrt[5]{32} = 2$

$\frac{dy}{dx} = \frac{1}{5} x^{-4/5}$ , $m = \frac{1}{5}(32)^{-4/5} = \frac{1}{80}$

$y = \frac{1}{80}(x-32) + 2$ , $\sqrt[5]{33} \approx \frac{1}{80}(33-32) + 2 = \boxed{2\frac{1}{80}}$

(b) Apply Newton's Method to the function $f(x) = x^5 - 33$, and compute $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ for $x_0 = 2$.

$f'(x) = 5x^4$

$f(2) = 2^5 - 33 = -1$

$f'(2) = 5(2^4) = 80$

$x_1 = 2 - \frac{-1}{80}$

$\boxed{x_1 = 2\frac{1}{80} = \frac{161}{80}}$

2. Let $x = \frac{t}{t+1}$ be the position of a moving particle, with $x$ in feet and the time $t$ in seconds. Find the time $t$ in the interval $1 \leq t \leq 7$ when the instantaneous velocity is equal to the average velocity on this interval.

$\frac{dx}{dt} = \frac{(t+1)(1) - t(1)}{(t+1)^2} = \frac{1}{(t+1)^2}$

| $t$ | $x$ |
|---|---|
| 1 | 1/2 |
| 7 | 7/8 |

$\frac{7/8 - 1/2}{7-1} = \frac{3/8}{6} = \frac{1}{16}$

$\boxed{\frac{1}{(t+1)^2} = \frac{1}{16}}$ , $(t+1)^2 = 16$, $t+1 = \pm 4$

or

$t^2 + 2t + 1 = 16$

etc.

$\boxed{t = 3 \text{ seconds}}$ or

$t = -5$ (not in interval)

Find the derivative of each function. Do not simplify.

3.

$$z = \sec(x^2)$$

$$\frac{dz}{dx} = \sec(x^2)\tan(x^2) \cdot (2x)$$

4.

$$x = 2^{\tan(\theta)}$$

$$\frac{dx}{d\theta} = 2^{\tan(\theta)} \cdot \ln(2) \cdot \sec^2(\theta)$$

5.

$$y = \frac{x\ln(x^2+3)}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\left((1)\ln(x^2+3) + x \cdot \frac{1}{x^2+3} \cdot (2x)\right) - x\ln(x^2+3) \cdot (2x)}{(x^2+1)^2}$$

6. Find an equation of the tangent line to the curve $x^2y^3 - x^3y^2 = 4$ at the point $(1, 2)$.

$$2xy^3 + 3x^2y^2\frac{dy}{dx} - 3x^2y^2 - 2x^3y\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx}(3x^2y^2 - 2x^3y) = 3x^2y^2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{3x^2y^2 - 2xy^3}{3x^2y^2 - 2x^3y}, \quad m = \frac{12-16}{12-4} = \frac{-4}{8} = \frac{-1}{2}$$

$$\boxed{y - 2 = \frac{-1}{2}(x-1)} \quad \text{or} \quad y = \frac{-1}{2}x + \frac{5}{2}$$

7. Find the $x-$ and $y-$coordinates of the absolute maximum of the function $y = x^3 - x^2 - x$ on the closed interval $0 \leq x \leq 2$.

$$\frac{dy}{dx} = 3x^2 - 2x - 1 = 0$$

$$(3x+1)(x-1) = 0$$

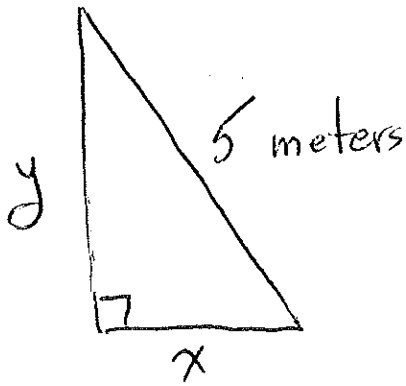
$$x = \frac{-1}{3} \quad \text{or} \quad \boxed{x = 1}$$

(not in interval)

| $x$ | $y = x^3 - x^2 - x$ |
|---|---|
| 0 | 0 |
| 1 | -1 |
| 2 | 2 |

The absolute maximum is at $(x, y) = (2, 2)$.

8. A ladder 5 meters long is leaning against a wall, and the bottom of the ladder slides along the floor away from the wall at a speed of 2 meters per second. Find the rate at which the top of the ladder slides down the wall at the instant when the bottom of the ladder is 3 meters from the wall.




$$x^2 + y^2 = 25$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = 2 \ \frac{\text{meters}}{\text{second}}$$

When $x = 3$ meters,

$$9 + y^2 = 25, \quad y^2 = 16,$$

$$y = 4 \text{ meters.}$$

$$2(3)(2) + 2(4)\frac{dy}{dt} = 0$$

$$8\frac{dy}{dt} = -12$$

$$\frac{dy}{dt} = \frac{-12}{8} = \boxed{\frac{-3}{2} \ \frac{\text{meters}}{\text{second}}}$$

9. Let $f(x) = 8x^{\frac{1}{3}} - x^{\frac{4}{3}}$. Note that $f'(x) = \frac{8}{3}x^{\frac{-2}{3}} - \frac{4}{3}x^{\frac{1}{3}}$ and $f''(x) = \frac{-16}{9}x^{\frac{-5}{3}} - \frac{4}{9}x^{\frac{-2}{3}}$.

(a) Find the $x$-coordinates of all critical points of $f(x)$.

Note $f'(x)$ is undefined for $\boxed{x=0.}$ Next, $f'(x)=0$ when $\frac{8}{3}x^{-2/3} = \frac{4}{3}x^{1/3}$, $2x^{-2/3} = x^{1/3}$, $\boxed{2=x}$. Or use

$$f'(x) = \frac{8}{3x^{2/3}} - \frac{4x^{1/3}}{3} = \frac{8-4x}{3x^{2/3}} = 0 \text{ for } 8-4x=0,\ x=2.$$

(b) Find the interval on which the function $f(x)$ is decreasing.

Either use "test values":

| $x$ | $-1$ | $1$ | $8$ (or use $x$ = billion) |
|---|---|---|---|
| $f'(x)$ | $\frac{12}{3}=4$ | $4/3$ | $\frac{2}{3}-\frac{8}{3}=-2$ |

or use $f'(x) = \frac{8-4x}{3x^{2/3}}$

| | | 0 | | 2 | |
|---|---|---|---|---|---|
| $8-4x$ | +++ | | +++ | | − − − |
| $3x^{2/3}$ | +++ | | +++ | | +++ |
| $f'(x)$ | ++ | 0 | ++ | 2 | − − |

So $f'(x)<0$ for $\boxed{x>2}$

(c) Find the interval on which the function $f(x)$ is concave up.

Note $f''(x)$ is undefined for $x=0$. Next, $f''(x)=0$ when $-\frac{16}{9}x^{-5/3} = \frac{4}{9}x^{-2/3}$, $-4x^{-5/3} = x^{-2/3}$, $-4=x$.

Either use "test values":

| $x$ | $-8$ | $-1$ | $1$ |
|---|---|---|---|
| $f''(x)$ | $\frac{1}{18}-\frac{1}{9}=\frac{-1}{18}$ | $\frac{12}{9}=\frac{4}{3}$ | $\frac{-20}{9}$ |

or use $f''(x) = \frac{-16-4x}{9x^{5/3}}$

| | | $-4$ | | 0 | |
|---|---|---|---|---|---|
| $-16-4x$ | +++ | | − − − | | − − − |
| $9x^{5/3}$ | − − − | | − − − | | +++ |
| $f''(x)$ | − − − ∩ | $-4$ | ++ ∪ | 0 | − − ∩ |

So $f''(x)>0$ for $\boxed{-4<x<0}$

10. The graph of the derivative $f'(x)$ of a function is given below.

(a) Find all critical numbers ($x$-coordinates).

$f'(x) = 0$ for $\boxed{x = -2 \text{ and } x = 3}$

(b) Where is the function $y = f(x)$ decreasing?

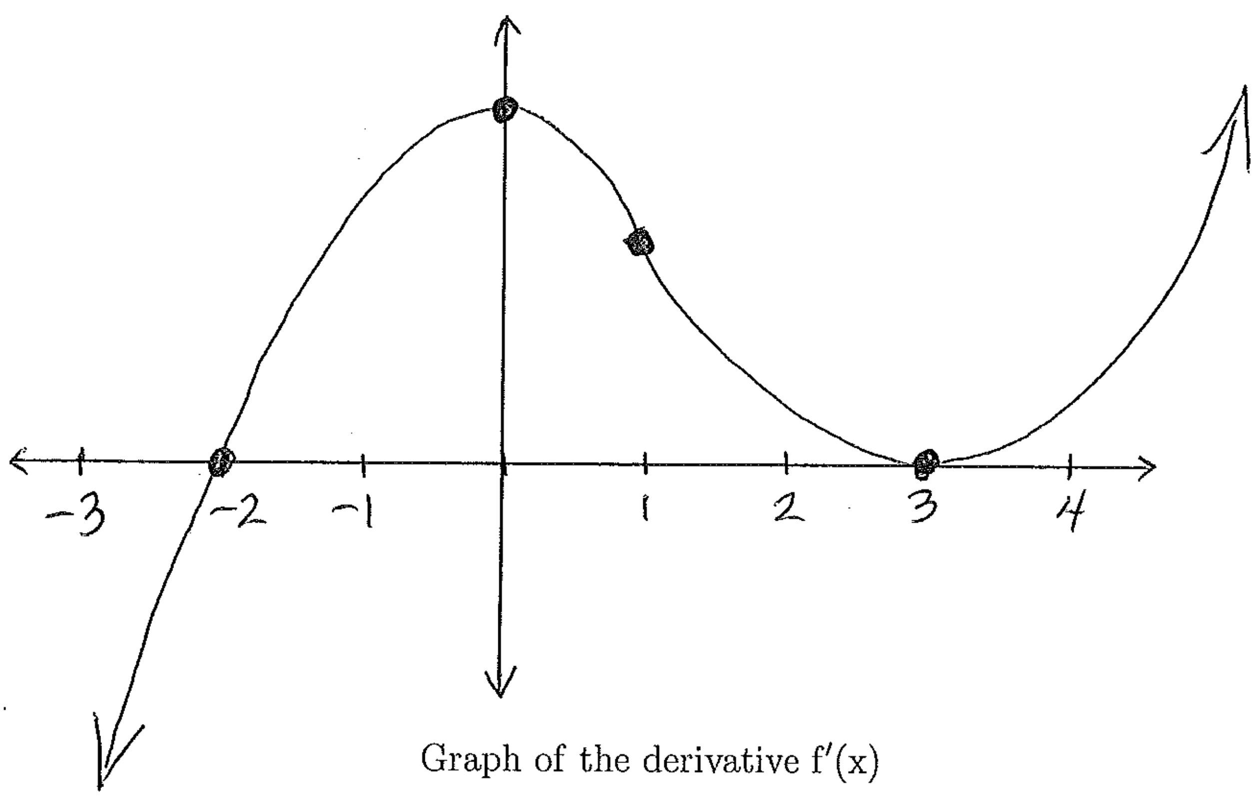
$f'(x) < 0$ for $\boxed{x < -2}$

(c) Where is the function $y = f(x)$ concave down?

$f'(x)$ is decreasing for $\boxed{0 < x < 3}$

(d) Find the $x$-coordinates of the inflection points.

The concavity changes for $\boxed{x = 0, \ x = 3}$




Graph of the derivative f'(x)