Math 220 Calculus I
Professor John Maginnis

Midterm Exam 3
November 19, 2015

Your name: Solutions

Rec. Instr.: ________ Rec. Time: ________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4	5
Points	/5	/5	/5	/5	/6
Problem	6	7	8	9	10
Points	/5	/5	/4	/8	/12

Evaluate the following integrals.

1.

$$\int (x^2+4)^2\,dx = \int (x^4 + 8x^2 + 16)\,dx =$$

$$\boxed{\frac{1}{5}x^5 + \frac{8}{3}x^3 + 16x + C}$$

2.

$$\int_1^2 2^t\,dt = \left[\frac{2^t}{\ln(2)}\right]_1^2 = \frac{4}{\ln(2)} - \frac{2}{\ln(2)}$$

$$= \boxed{\frac{2}{\ln(2)}}$$

3.

$$\int \tan^3(x)\sec^2(x)\,dx = \int u^3\,du = \frac{1}{4}u^4 + C$$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

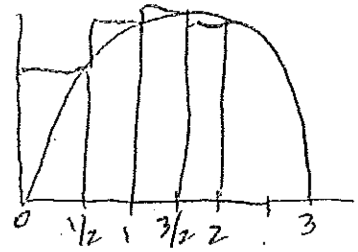
$$= \boxed{\frac{1}{4}\tan^4(x) + C}$$

4. Approximate the area under the curve $y = 12x - 4x^2$ between $x = 0$ and $x = 2$ using four rectangles and the right endpoint method.

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

x	$y = 12x - 4x^2$	
0	0	
$1/2$	$6-1=5$	right endpoints
1	$12-4=8$	right endpoints
$3/2$	$18-9=9$	right endpoints
2	$24-16=8$	right endpoints

$$A \approx 5 \times \frac{1}{2} + 8 \times \frac{1}{2} + 9 \times \frac{1}{2} + 8 \times \frac{1}{2} = \frac{30}{2} = \boxed{15}$$



5. A ball thrown vertically from the roof of a building 150 feet tall hits the ground 3 seconds later. Was the ball thrown upward or downward? With what speed was it thrown? (Recall that the acceleration of gravity is $a = -32$ feet per second per second.)

$a = -32$, $a = \frac{dv}{dt}$, $v = \int a\,dt$

$v = \int -32\,dt = -32t + C$. Since $v = \frac{dy}{dt}$,

$y = \int v\,dt = \int (-32t + C)\,dt = -16t^2 + Ct + C'$.

When $t = 0$, $y = 150 = 0 + 0 + C'$, so $C' = 150$.

$y = -16t^2 + Ct + 150$. When $t = 3$, $y = 0$, so

$0 = -144 + 3C + 150 = 3C + 6$, so

$C = -2 \frac{\text{feet}}{\text{sec}}$. Note when $t = 0$, $v = -32(0) + C = C$,

so at time $t = 0$ the ball was thrown downward ($C < 0$) at a speed of $2 \frac{\text{feet}}{\text{sec}}$ (or velocity of $-2 \frac{\text{feet}}{\text{sec}}$).

6. Define $F(x) = \int_1^x \left(\frac{\sin(\frac{\pi t}{6})}{t^2}\right) dt$. Find an equation of the tangent line to $y = F(x)$ at $x = 1$.

The Fundamental Theorem of Calculus tell us that

$F'(x) = \dfrac{\sin\left(\frac{\pi x}{6}\right)}{x^2}$, so the slope $m = F'(1) = \dfrac{\sin\left(\frac{\pi}{6}\right)}{1} = \boxed{\frac{1}{2}}$.

Also when $x = 1$, $y = F(1) = \int_1^1 \dfrac{\sin\left(\frac{\pi t}{6}\right)}{t^2} dt = 0$ (there is no area between $t=1$ and $t=1$). Thus $\boxed{y = \frac{1}{2}(x-1).}$

7. An object moves along the x-axis with velocity $v = 12t^3 - 12t^2$ centimeters per second. Find the total distance traveled for the interval $-1 \leq t \leq 2$ seconds.

Note $v = 12t^3 - 12t^2 = 12t^2(t-1) = 0$ for $t = 0$, $t = 1$, and $v \geq 0$ for $t \geq 1$, and $v \leq 0$ for $t \leq 1$.

$$\int_{-1}^{1} \underbrace{-(12t^3 - 12t^2)}_{|v| = -v \text{ when } v \leq 0} dt + \int_1^2 \underbrace{(12t^3 - 12t^2)}_{|v| = v \text{ when } v \geq 0} dt$$

$$= \left[-3t^4 + 4t^3\right]_{-1}^{1} + \left[3t^4 - 4t^3\right]_1^2$$

$$= (1 - (-7)) + (\underset{48-32}{\underset{\shortparallel}{16}} - (-1))$$

$$= 8 + 17 = \boxed{25 \text{ centimeters}}$$

Remarks:

t	$3t^4 - 4t^3$
-1	7
0	0
1	-1
2	$16 = 48 - 32$

Then $8 = 7 - (-1)$, $17 = 16 - (-1)$

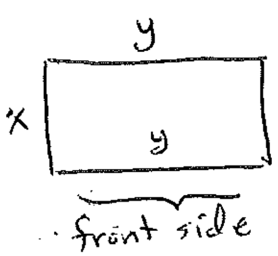
8. Assume that $\int_0^6 f(x)dx = 9$, $\int_4^6 f(x)dx = 5$, and $\int_0^4 g(x)dx = 8$. Compute $\int_0^4 (5f(x) - 3g(x))dx$.

Note $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx$, so that

$\int_0^4 f(x)\,dx = 9 - 5 = 4$. Then $\int_0^4 (5f(x) - 3g(x))\,dx =$

$$5 \cdot \int_0^4 f(x)\,dx - 3\int_0^4 g(x)\,dx = 5(4) - 3(8) = 20 - 24 = \boxed{-4}$$

9. A landscaper is designing a fence along the four sides of a rectangular garden, which is to have an area of 5000 square feet. The fencing material for three sides costs \$ 10 per foot, but the fencing along the front side of the garden will cost \$ 30 per foot. Find the length and width of the garden in order to minimize the total cost. Verify that your answer is a minimum.



$A = xy$, $5000 = xy$, $\boxed{y = \frac{5000}{x}}$.

$C = 10(x + y + x) + 30(y) = 20x + 40y$.

$$C = 20x + 40\left(\frac{5000}{x}\right) = \boxed{20x + \frac{200{,}000}{x}}$$

$$\frac{dC}{dx} = 20 - \frac{200{,}000}{x^2} = 0$$

$20x^2 = 200{,}000$

$x^2 = 10{,}000$

$x = \sqrt{10{,}000} = \boxed{100 \text{ feet}}$

$y = \frac{5000}{x} = \frac{5000}{100} = \boxed{50 \text{ feet}}$

Note $C'' = \frac{400{,}000}{x^3}$

$C''(100) = \frac{4}{10} > 0$, $++$

So $x = 100$ is a local minimum

(or use 1^{st} derivative test)

10.

Let $y = f(x) = \left(\dfrac{x-2}{x}\right)^2$. You will graph this function on the next page.

Use the derivatives : $f'(x) = \dfrac{4(x-2)}{x^3}$ and $f''(x) = \dfrac{-8(x-3)}{x^4}$.

(a) Find the interval where the function $y = f(x)$ is decreasing. Show all work.

Note $x = 2$ is the critical number, and $x = 0$ is an asymptote

Test values

x	$f'(x)$
-1	3
1	-4
3	$4/27$

or use :

	$x<0$		$0<x<2$		$x>2$
$4(x-2)$	$--$		$--$		$++$
x^3	$--$		$++$		$++$
		0		2	
$f'(x)$	$++$		$--$		$++$

$f'(x) < 0$ for $\boxed{0 < x < 2.}$

(b) Find the interval where the function $y = f(x)$ is concave down. Show all work.

Note $x = 3$ is a possible inflection point, and $x = 0$ is an asymptote.

Test values

x	$f''(x)$
-1	32
1	16
4	$-1/32$

or use :

	$x<0$		$0<x<3$		$x>3$
$-8(x-3)$	$++$		$++$		$--$
x^4	$++$		$++$		$++$
		0		3	
$f''(x)$	$++$		$++$		$--$

$f''(x) < 0$ for $\boxed{x > 3.}$

(c) Find the intercept and all asymptotes of $y = \left(\frac{x-2}{x}\right)^2$.

$y=0$ when $x=2$. The x-intercept is $(2,0)$.
There is a vertical asymptote $x=0$, and a horizontal asymptote $y=1$ since $\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^2 = \left(\lim_{x\to\infty}\frac{x-2}{x}\right)^2 = \left(\lim_{x\to\infty}\frac{1-\frac{2}{x}}{1}\right)^2 = \left(\frac{1-0}{1}\right)^2 = 1.$

(d) Graph the function $y = \left(\frac{x-2}{x}\right)^2$, clearly labeling the minimum and the inflection point.