Math 220 Calculus I
Professor John Maginnis

Final Exam
December 16, 2015

Your name: Solutions

Rec. Instr.: ______________________ Rec. Time: ______________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 120 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4	5	6
Points	/6	/8	/6	/8	/5	/7
Problem	7	8	9	10	11	12
Points	/10	/7	/7	/8	/8	/8
Problem	13	14	15	16		Total
Points	/8	/8	/8	/8		/120

1. Find the derivative $\frac{dz}{dx}$.

$z = x^4 \cdot \arctan(5x)$

$$z' = \frac{dz}{dx} = 4x^3 \cdot \arctan(5x) + x^4 \cdot \left(\frac{1}{(5x)^2+1}\right)(5)$$

$$= \boxed{4x^3 \cdot \arctan(5x) + \frac{5x^4}{25x^2+1}}$$

2. Find the interval where the function $y = f(x)$ is increasing.

$y = f(x) = x^4 - 4x^3$

$$y' = \frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \text{ for}$$

$$x = 0, \quad x = 3$$

	$x<0$	$0<x<3$	$x>3$
$4x^2$	$++$	$++$	$++$
$x-3$	$--$	$--$	$++$
$f'(x)$	$--$	$--$	$++$

Test Values

x	$f'(x)$
-1	-16
1	-8
4	64

$$f'(x) > 0 \text{ for } \boxed{x > 3}$$

3. Evaluate the limit.

$$\lim_{x \to -1} \frac{2x^2 - x - 3}{5x^2 + 9x + 4} = \frac{2+1-3}{5-9+4} = \frac{0}{0}, \text{ not determined.}$$

$$= \lim_{x \to -1} \frac{(x+1)(2x-3)}{(x+1)(5x+4)} = \lim_{x \to -1} \frac{2x-3}{5x+4}$$

$$= \frac{-2-3}{-5+4} = \frac{-5}{-1} = \boxed{5}$$

4. Find the interval where the function $y = f(x)$ is concave down.

$y = f(x) = x^4 - 2x^3 - 12x^2$

$$\frac{dy}{dx} = f'(x) = 4x^3 - 6x^2 - 24x$$

$$\frac{d^2y}{dx^2} = f''(x) = 12x^2 - 12x - 24 = 12(x^2 - x - 2) =$$

$$12(x-2)(x+1) = 0 \text{ for } x = -1, 2.$$

		-1		2	
$x-2$	$--$		$--$		$++$
$x+1$	$--$		$++$		$++$
$f''(x)$	$++$		$--$		$++$

Test values

x	$f''(x)$
-2	48
0	-24
3	48

$$f''(x) < 0 \text{ for } \boxed{-1 < x < 2}$$

5. Compute the right hand limit.

$$\lim_{x \to -5^+} \frac{x-5}{x+5} = \frac{-5-5}{-5+5} = \frac{-10}{0} = \pm\infty.$$

Since $x > -5$, $x+5 > 0$, and so

$$\lim_{x \to -5^+} \frac{x-5}{x+5} = \frac{-10}{\text{small positive number}} = \boxed{-\infty}$$

6. Find the x- and y-coordinates of the absolute maximum of the function on the interval $-2 \leq x \leq 2$.

$$y = \frac{x}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{(1-x)(1+x)}{(x^2+1)^2} = 0 \text{ for } x = \pm 1.$$

x	$y = \frac{x}{x^2+1}$
-2	$-2/5$
-1	$-1/2$
1	$1/2$ = absolute maximum at $(1, \frac{1}{2})$
2	$2/5$

Note $\frac{1}{2} = \frac{5}{10} > \frac{4}{10} = \frac{2}{5}$.

7. Approximate the square root of three using the following two methods.

(a) Find an equation of the tangent line to the curve $y = \sqrt{x}$ at $x_0 = 4$, and use this to approximate $\sqrt{3}$.

$$y = \sqrt{x} = x^{1/2} \qquad \text{Note } \sqrt{4} = 2.$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$m = y'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}, \text{ so } \boxed{y - 2 = \frac{1}{4}(x-4)}.$$

$$\text{Then } \sqrt{3} \approx 2 + \frac{1}{4}(3-4) = 2 - \frac{1}{4} = \boxed{\frac{7}{4}} = 1.75$$

(b) Apply Newton's Method to the function $f(x) = x^2 - 3$, and compute both $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ for $x_0 = 2$, and $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$f'(x) = 2x \qquad f(2) = 4 - 3 = 1$$

$$f'(2) = 4$$

$$x_1 = 2 - \frac{1}{4} = \boxed{\frac{7}{4}} = 1.75$$

$$f'\left(\frac{7}{4}\right) = 2\left(\frac{7}{4}\right) = \frac{7}{2}, \quad f\left(\frac{7}{4}\right) = \frac{49}{16} - 3 = \frac{1}{16}$$

$$x_2 = \frac{7}{4} - \frac{1/16}{7/2} = \frac{7}{4} - \frac{1}{56} = \frac{98}{56} - \frac{1}{56} = \boxed{\frac{97}{56}}$$

8. Find the derivative $\frac{dy}{dx}$ if $xy^2 - y = 6$.

$$(1)y^2 + 2xy\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2xy-1) = -y^2$$

$$\boxed{\frac{dy}{dx} = \frac{-y^2}{2xy-1}}$$

9. Use the method of substitution to evaluate the indefinite integral.

$$\int \frac{2^t}{1+2^t}\,dt$$

Let $u = 1+2^t$

Then $du = 2^t \ln(2)\,dt$

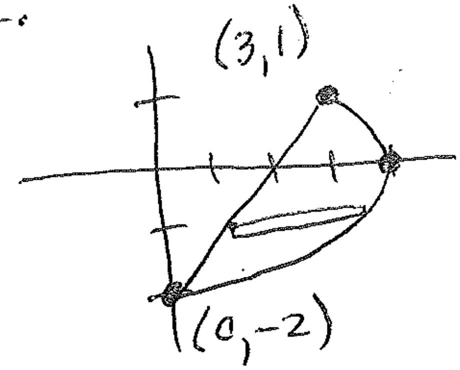
$$\frac{du}{\ln(2)} = 2^t\,dt$$

$$\int \frac{1}{u}\cdot\frac{du}{\ln(2)} = \frac{1}{\ln(2)}\int\frac{du}{u} = \frac{\ln|u|}{\ln(2)} + C$$

$$= \boxed{\frac{\ln(1+2^t)}{\ln(2)} + C}$$

10. Find the area between the line $y = x - 2$ and the parabola $x = 4 - y^2$.

Note $x = y + 2.$



Intersection Points

$4 - y^2 = y + 2$

$0 = y^2 + y - 2 = (y-1)(y+2)$

$y = 1, \quad y = -2$

$x = 3, \quad x = 0$

$$A = \int_{-2}^{1} (4 - y^2) - (y+2)\, dy$$

$$= \int_{-2}^{1} 2 - y - y^2 \, dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

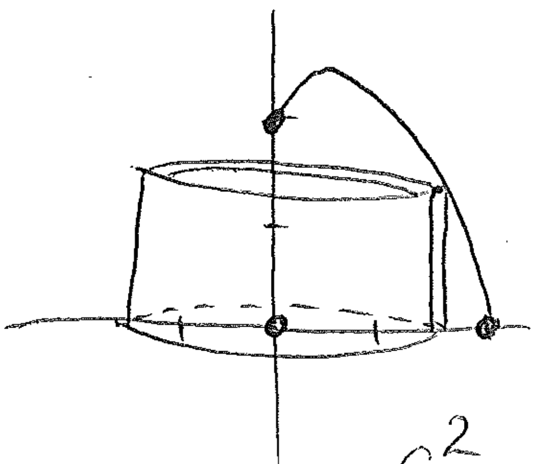
$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right) = 4\frac{1}{2} = \boxed{\frac{9}{2}}$$

Note $A = \displaystyle\int_0^3 \left(x - 2 + \sqrt{4-x}\right) dx + \int_3^4 2\sqrt{4-x}\, dx =$

$\frac{19}{6} + \frac{4}{3} = \frac{27}{6} = \frac{9}{2}.$

11. Use the method of cylindrical shells to find the volume of revolution formed by revolving the region in the first quadrant bounded by $x = 0$, $y = 0$, and $y = -x^2 + x + 2$, around the y-axis.

If $x=0$ then $y=2$. If $y=0=-x^2+x+2$ then $x^2-x-2=0$, $(x-2)(x+1)=0$, so $x=2$ or $x=-1$. (Note $x=-1$ is not in first quadrant)



Cylindrical Shells

$$dV = 2\pi R H \, dx$$

$$R = x, \quad H = y = -x^2+x+2$$

$$V = 2\pi \int_0^2 x(-x^2+x+2)\,dx$$

$$V = 2\pi \int_0^2 -x^3+x^2+2x \, dx$$

$$V = 2\pi \left[\frac{-x^4}{4} + \frac{x^3}{3} + x^2 \right]_0^2 = 2\pi \left[-4 + \frac{8}{3} + 4 - (0) \right]$$

$$V = \boxed{\frac{16\pi}{3}}$$

12. The driver of a car who is driving at a speed of 60 $\frac{\text{feet}}{\text{second}}$ applies the brakes when he sees a tree 100 feet away that has fallen on the road. The brakes apply a constant deceleration of $-20\ \frac{\text{feet}}{\text{second}^2}$. How far does the car travel before it stops? Does the car hit the tree?

$a = -20$, $a = \frac{dv}{dt}$

$v = \int -20\, dt = -20t + C$

Let $t=0$ denote the instant at which the driver sees the tree and applies the brakes.

Then $v = 60 = -20(0) + C$, so $C = 60$.

$v = -20t + 60$, $v = \frac{dx}{dt}$

$x = \int (-20t + 60)\, dt = -10t^2 + 60t + C'$

Let $x=0$ denote the position when $t=0$. Then $C'=0$.

The car stops when $v=0$, $-20t+60=0$,

$t = 3$ seconds.

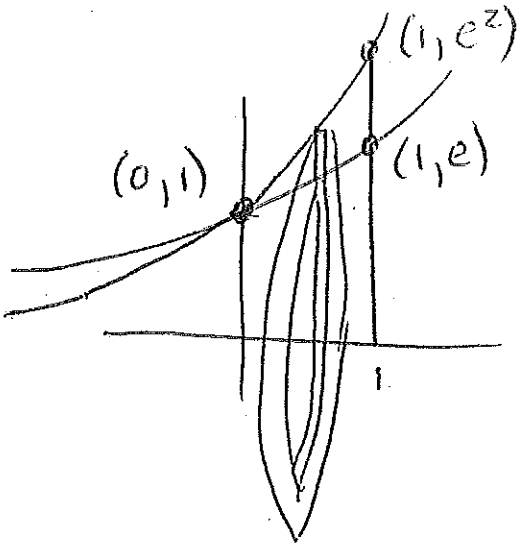
$x(3) = -10(9) + 60(3) = -90 + 180 = \boxed{90 \text{ feet}}$

The car does not hit the tree, $90 < 100$.

or $\int_0^3 (-20t+60)\, dt = 90$ feet.

13. Find the volume of revolution formed by revolving the region bounded by $y = e^x$, $y = e^{2x}$ and $x = 1$, around the x-axis. (Note that the two exponential curves intersect when $x = 0$. Your final answer will be written in terms of the numbers π and e.)

<u>Washers:</u>



$$dV = \pi\left(R^2 - r^2\right)dx$$

$$R = e^{2x}, \quad r = e^x$$

$$V = \pi \int_0^1 \left(e^{2x}\right)^2 - \left(e^x\right)^2 \, dx$$

$$V = \pi \int_0^1 e^{4x} - e^{2x} \, dx = \pi \left[\frac{e^{4x}}{4} - \frac{e^{2x}}{2}\right]_0^1$$

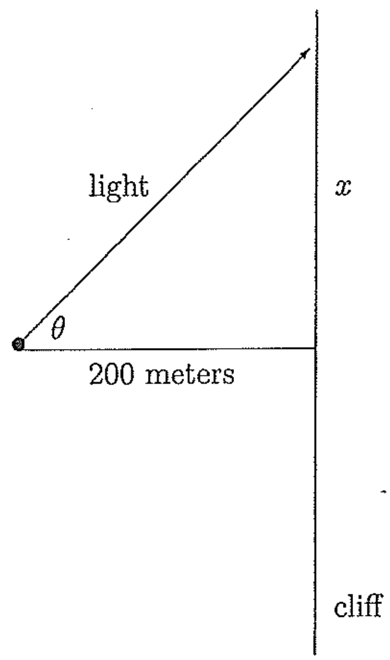
$u = 4x$
$du = 4dx$
$\frac{1}{4}du = dx$

$v = 2x$
$dv = 2dx$
$\frac{1}{2}dv = dx$

$$\boxed{= \pi\left(\frac{e^4}{4} - \frac{e^2}{2}\right) - \pi\left(\frac{1}{4} - \frac{1}{2}\right)}$$

$$= \boxed{\frac{\pi\left(e^4 - 2e^2 + 1\right)}{4}}$$

14. A lighthouse sits on a small island 200 meters away from a long straight cliff. The beam of light makes one full revolution every minute, shining on the cliff for some of that time. Find the speed at which the spot of light moves along the cliff at the instant when the beam of light forms a 45 degree angle with the line perpendicular to the cliff.



$$\tan(\theta) = \frac{x}{200}$$

$$x = 200\tan(\theta)$$

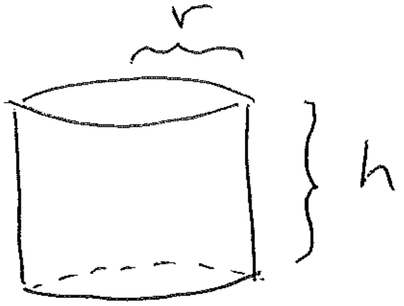
$$\frac{dx}{dt} = 200\sec^2(\theta)\frac{d\theta}{dt}$$

one full revolution every minute, so $\frac{d\theta}{dt} = \frac{2\pi \text{ radians}}{\text{minute}}$

When $\theta = 45^\circ = \frac{\pi}{4}$, $\cos(\theta) = \frac{\sqrt{2}}{2}$, $\sec(\theta) = \sqrt{2}$, $\sec^2(\theta) = 2$

Then $\frac{dx}{dt} = 200(2)(2\pi) = \boxed{800\pi \frac{\text{meters}}{\text{minute}}}$

15. A cylinder is to have a total surface area of 6π square meters, including the top, bottom, and sides. Find the largest possible volume of such a cylinder. Verify that your answer is a maximum.



$$A = \underbrace{2\pi r^2}_{\text{top and bottom}} + \underbrace{2\pi r h}_{\text{sides}} = 6\pi$$

$$h = \frac{6\pi - 2\pi r^2}{2\pi r} = \frac{3}{r} - r$$

$$V = \pi r^2 h = \pi r^2\left(\frac{3}{r} - r\right)$$

$$V = 3\pi r - \pi r^3$$

$$V' = 3\pi - 3\pi r^2 = 0$$

$$r^2 = 1, \quad \boxed{r = 1 \text{ meter}}$$

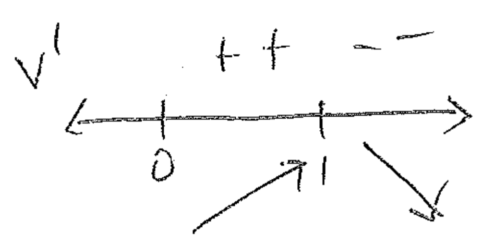
$$V = 3\pi - \pi = \boxed{2\pi \text{ cubic meters}}$$

To <u>verify</u>, <u>either</u> use $V'' = -6\pi r$

$$V''(1) = -6\pi < 0$$

$r = 1$ is a local <u>maximum</u>.

<u>or</u>



<u>or</u> closed interval $0 \le r \le \sqrt{3}$,

r	V
0	0
1	2π = absolute <u>maximum</u>.
$\sqrt{3}$	0 ($h=0$)

16. The graph of a function $y = f(t)$ is given below. Define another function $g(x) = \int_1^x f(t)\,dt$.

(a) Find $g(1)$. $= \int_1^1 f(t)\,dt = \boxed{0}$ since there is no area between $t=1$ and $t=1$

(b) Find $g'(1)$. $= f(1) = \boxed{2}$ (the point $(1,2)$ on graph)

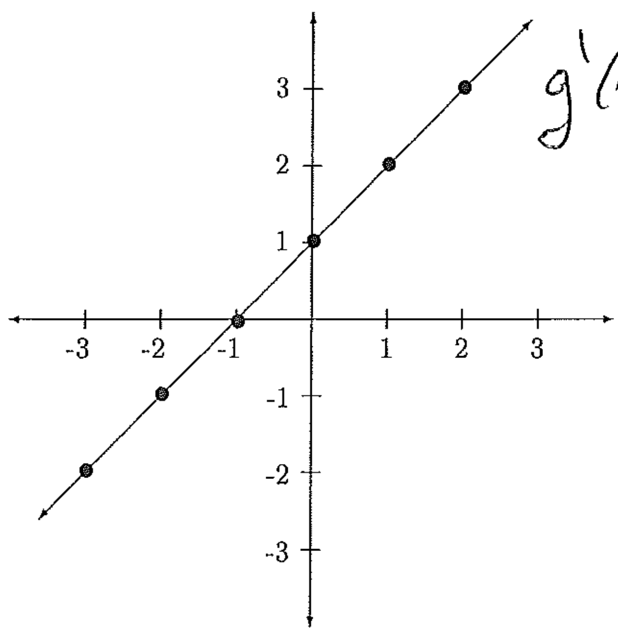
The Fundamental Theorem of Calculus says $g'(x) = f(x)$.

(c) Find $g''(1)$. $= f'(1) =$ slope of graph $= \boxed{1}$

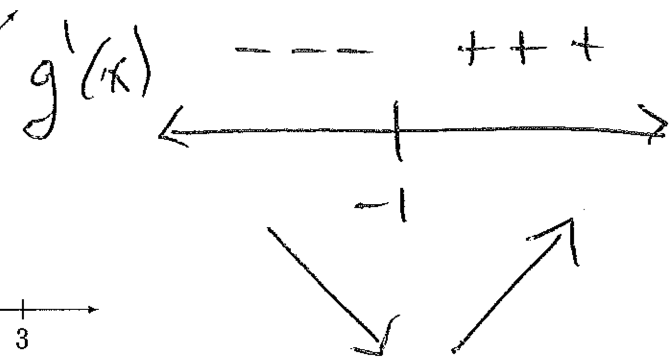
Since $g'(x) = f(x)$, we have $g''(x) = f'(x)$.

(d) Find the x-coordinate of the minimum of $g(x)$.

$\boxed{x = -1}$ For $t < -1$, $f(t) < 0$ (below axis)
For $t > -1$, $f(t) > 0$ (above axis)



The graph of $y = f(t)$.



minimum at $x = -1$.