Name Solutions Rec. Instr.

Signature Rec. Time

Math 220

Final Exam

May 13, 2015

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		16	8		5
2		8	9		2
3		6	10		6
4		18	11		6
5		12	12		6
6		6	13		4
7		5	Total Score		100

1. (4 points each) Evaluate the following:

A. $\lim_{x \to -2} \frac{x+2}{x^2-4} =$ $\lim_{x \to -2} \frac{x+2}{(x+2)(x-2)} = \lim_{x \to -2} \frac{1}{x-2}$

$= \frac{1}{-2-2} = -\frac{1}{4}$

B. $\lim_{x \to \infty} \frac{2x^4 - 3x^3 + x + 9}{7x^4 + 2x^2 - 3} =$ $\frac{2}{7}$

C. $\frac{d}{dx} \int_0^x \cos(e^t)\, dt =$ $\cos(e^x)$

D. $\frac{d}{dx}\left(\frac{2^x \cdot \tan(x)}{5x+2}\right) =$ $\frac{\left[\frac{d}{dx} 2^x \cdot \tan(x)\right](5x+2) - 2^x \cdot \tan(x)\left[\frac{d}{dx}(5x+2)\right]}{(5x+2)^2}$

$= \frac{\left[2^x \cdot \ln(2) \cdot \tan(x) + 2^x \cdot \sec^2(x)\right](5x+2) - 2^x \cdot \tan(x) \cdot 5}{(5x+2)^2}$

2. (8 points) If a factory spends L thousand dollars on labor and M thousand dollars on materials, it will produce LM^2 thousand watches. In order for the factory to produce four thousand watches, how much should the company spend on labor and materials in order to minimize the total cost $L + M$? (Justify why your answer corresponds to an absolute minimum.)

Minimize Cost $C = L + M$.

We have $LM^2 = 4$ so $L = \frac{4}{M^2}$.

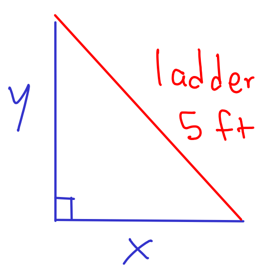
Minimize $C(M) = \frac{4}{M^2} + M$ on $(0, \infty)$.

$C'(M) = -\frac{8}{M^3} + 1$. $C'(M) = 0 \iff 1 = \frac{8}{M^3} \iff M^3 = 8 \iff M = 2$.

1st Derivative	2nd Derivative
Sign of $C'(M)$: $-$ on $(0, 2)$, $+$ after 2	$C''(M) = \frac{24}{M^4} > 0$ on $(0, \infty)$ $C(M)$ is concave up on $(0, \infty)$

The cost is minimized when $M = 2$ (\$2000 are spent on materials) and $L = \frac{4}{2^2} = 1$ (\$1000 is spent on labor).

3. (6 points) A 5-foot ladder rests against the wall. The bottom of the ladder slides away from the wall at a rate of 2 feet/second. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 feet from the wall?



Want: $\frac{dy}{dt}$ when $x = 3$ ft.

Know: $\frac{dx}{dt} = 2$ ft/s.

$x^2 + y^2 = 5^2$ so $\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt} 5^2$.

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow 2y\frac{dy}{dt} = -2x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = -\frac{x \cdot \frac{dx}{dt}}{y}$

When $x = 3$ ft, $3^2 + y^2 = 5^2 \Rightarrow y^2 = 5^2 - 3^2 \Rightarrow y = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$ ft.

Then, $\frac{dy}{dt} = \frac{-3 \cdot 2}{4} = -\frac{3}{2}\ \frac{\text{ft}}{\text{s}}$.

4. (6 points each) Evaluate the following:

A. $\int_0^1 \sqrt{x}\, dx =$ $\frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}\cdot 1^{3/2} - \frac{2}{3}\cdot 0^{3/2}$

$= \frac{2}{3}$

B. $\int e^x \sin(e^x)\, dx =$ $\int \sin(u)\,du = -\cos(u) + C = -\cos(e^x) + C$

Let $u = e^x$.

$\frac{du}{dx} = e^x$

$du = e^x dx$

C. $\int_1^e \frac{(\ln(x))^3}{x}\, dx =$ $\int_0^1 u^3\,du = \frac{u^4}{4}\Big|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$

Let $u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx$

x	u	
e	1	$\leftarrow \ln(e) = 1$
1	0	$\leftarrow \ln(1) = 0$

5. (6 points each) Find $\dfrac{dy}{dx}$ for:

A. $x^3 + xy + y^3 = 10$

$$\frac{d}{dx}\left[x^3+xy+y^3\right]=\frac{d}{dx}[10]$$

$$3x^2+1\cdot y+x\cdot\frac{dy}{dx}+3y^2\cdot\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}+3y^2\frac{dy}{dx}=-3x^2-y$$

$$(x+3y^2)\frac{dy}{dx}=-3x^2-y$$

$$\frac{dy}{dx}=\frac{-3x^2-y}{x+3y^2}$$

B. $y = \dfrac{x^x}{(3x+1)^5}$

$$\ln(y)=\ln\left(\frac{x^x}{(3x+1)^5}\right)=\ln(x^x)-\ln(3x+1)^5=x\cdot\ln(x)-5\ln(3x+1)$$

$$\frac{d}{dx}\ln(y)=\frac{d}{dx}\left[x\cdot\ln(x)-5\cdot\ln(3x+1)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=1\cdot\ln(x)+x\cdot\frac{1}{x}-5\cdot\frac{3}{3x+1}=\ln(x)+1-\frac{15}{3x+1}$$

$$\frac{dy}{dx}=y\left(\ln(x)+1-\frac{15}{3x+1}\right)=\frac{x^x}{(3x+1)^5}\cdot\left(\ln(x)+1-\frac{15}{3x+1}\right)$$

6. (6 points) Let $f(x) = x^2 + 1$. Using the **limit definition of the derivative**, find $f'(3)$.

$$f'(3)=\lim_{h\to0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to0}\frac{((3+h)^2+1)-(3^2+1)}{h}$$

$$=\lim_{h\to0}\frac{9+6h+h^2+1-9-1}{h}=\lim_{h\to0}\frac{6h+h^2}{h}$$

$$=\lim_{h\to0}(6+h)=6+0=6$$

7. **A.** (3 points) Find the linearization of $h(x) = \ln(x)$ at $x = 1$.

$h'(x) = \frac{1}{x}$. The linearization of $h(x)$ at $x=1$ is

$L(x) = h(1) + h'(1)(x-1) = \ln(1) + \frac{1}{1}(x-1) = x-1$

B. (2 points) Use the linearization from Part A to estimate $\ln(1.1)$.

$\ln(1.1) = h(1.1) \approx L(1.1) = 1.1 - 1 = .1$

1.1 is close to 1

8. For $0 \le t \le 20$, let $w(t) = 20 - t$ be the rate that water flows out of a storage tank in gallons per minute at time t minutes after the tank ruptures.

A. (3 points) Calculate $\int_0^{10} w(t)\,dt$.

$$\int_0^{10} w(t)\,dt = \int_0^{10} (20-t)\,dt = \left(20t - \frac{t^2}{2}\right)\Big|_0^{10}$$
$$= \left(20\cdot 10 - \frac{10^2}{2}\right) - \left(20\cdot 0 - \frac{0^2}{2}\right)$$
$$= 200 - 50 = 150$$

B. (2 points) What does $\int_0^{10} w(t)\,dt$ represent?

By the Net Change Theorem, this is the number of gallons that flow out of the tank within the first ten minutes after the tank ruptures.

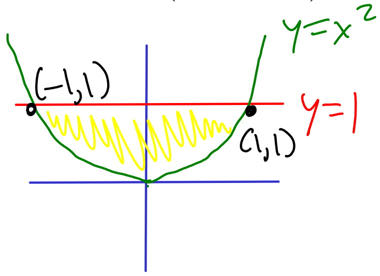
9. (1 point each) Let $p(x) = x^2 - 1$. Over what intervals are $p(x)$:

$p'(x) = 2x \quad p''(x) = 2$

A. Concave up: $(-\infty, \infty)$

B. Concave down: nowhere

10. (6 points) Find the area bounded between $y = x^2$ and $y = 1$.



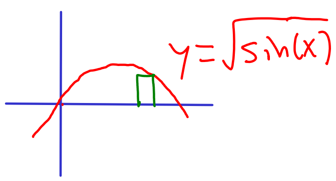
$x^2 = 1$ when $x = \pm 1$.

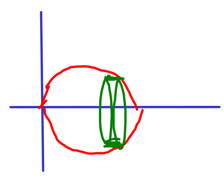
$$\text{AREA} = \int_{-1}^{1} |x^2 - 1|\, dx = \int_{-1}^{1} (1 - x^2)\, dx$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \left(1 - \frac{1^3}{3}\right) - \left((-1) - \frac{(-1)^3}{3}\right)$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$

11. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{\sin(x)}$ from 0 to π around the x-axis.





$$\text{Volume} = \int_0^{\pi} \pi \cdot \left[\sqrt{\sin(x)}\right]^2 dx = \int_0^{\pi} \pi \cdot \sin(x)\, dx$$

$$= -\pi \cos(x)\Big|_0^{\pi} = -\pi\cos(\pi) + \pi\cos(0)$$

$$= \pi + \pi = 2\pi$$

12. (6 points) Find the absolute maximum and absolute minimum of $v(x) = x + \sin(x)$ on the interval $[0, 2\pi]$.

$v'(x) = 1 + \cos(x)$ is defined on $[0, 2\pi]$.

$v'(x) = 0$ when $\cos(x) = -1$, which happens when $x = \pi + 2\pi k$ (k an integer).

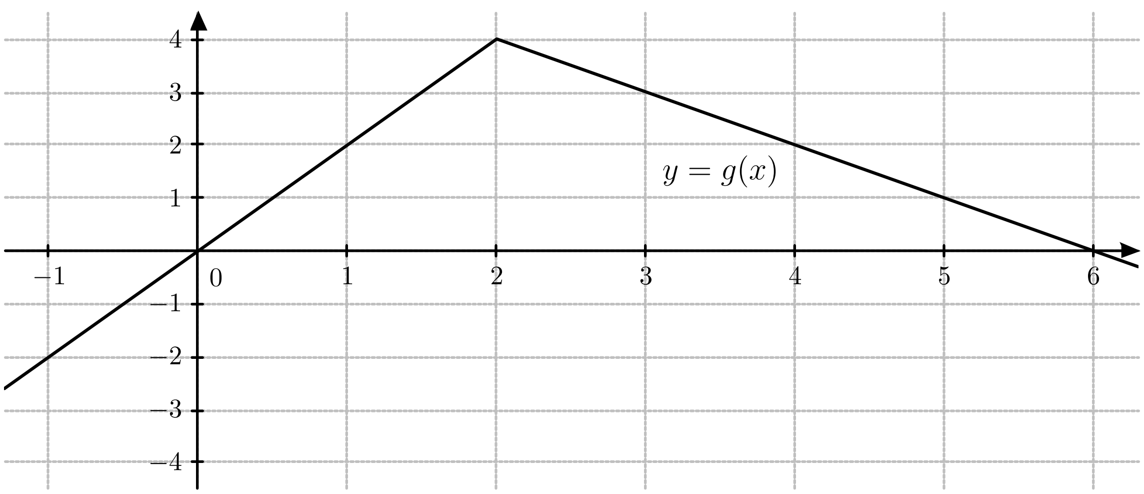
The only critical point in $[0, 2\pi]$ is at $x = \pi$.

$v(2\pi) = 2\pi + \sin(2\pi) = 2\pi$

$v(\pi) = \pi + \sin(\pi) = \pi$

$v(0) = 0 + \sin(0) = 0$

On $[0, 2]$, $v(x)$ has an absolute max at $(2\pi, 2\pi)$ and an absolute minimum at $(0, 0)$.



13. (2 points each) $y = g(x)$ is plotted above. Evaluate the following definite integrals.

A. $\int_{-1}^{2} g(x)\,dx = \frac{1}{2} \cdot 2 \cdot 4 - \frac{1}{2} \cdot 1 \cdot 2 = 4 - 1 = 3$

B. $\int_{2}^{6} (g(x) + 4)\,dx = \int_2^6 g(x)\,dx + \int_2^6 4\,dx = \frac{1}{2} \cdot 4 \cdot 4 + 4 \cdot 4$

$= 8 + 16 = 24$