Name Solutions Rec. Instr.

Signature Rec. Time

Math 220

Final Exam

December 14, 2016

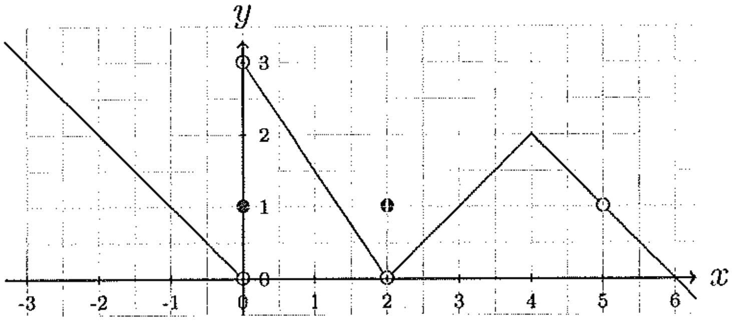
No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 1 hour and 50 minutes to complete the exam.

Total = 200 points. **Show your work unless stated otherwise.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		10	9		12
2		18	10		10
3		8	11		24
4		24	12		8
5		12	13		12
6		10	14		8
7		10	15		8
8		10	16		16

Total Score:

1. (2 points each) Evaluate the following for the graph below or state they do not exist. No work needs to be shown.



a. Find $\lim_{x \to 0^+} f(x) = 3$

b. Find $\lim_{x \to 2} f(x) = 0$

c. Indicate all values of x at which $f'(x)$ is not defined. $x = 0, 2, 4, 5$

d. Indicate all values of x at which $f(x)$ is not continuous. $x = 0, 2, 5$

e. Find $f'(1) =$ slope at $x=1$ $= \frac{\Delta y}{\Delta x} = \frac{-3}{2} = -\frac{3}{2}$

2. (6 points each) Evaluate the following limits.

a. $$\lim_{x \to 3} \frac{x-3}{9x - x^3} = \lim_{x \to 3} \frac{x-3}{x(9-x^2)} = \lim_{x \to 3} \frac{\cancel{x-3}(-1)}{x(\cancel{3-x})(3+x)} = \frac{-1}{3 \cdot 6} = -\frac{1}{18}$$

$$\underline{\underline{\text{or}}} \text{ by L'Hopital} = \lim_{x \to 3} \frac{1}{9-3x^2} = \frac{1}{9-27} = -\frac{1}{18}$$

b. $$\lim_{h \to 0} \frac{\tan(2h)}{\sin(5h)} = \lim_{h \to 0} \frac{2\sec^2(2h)}{5\cos(5h)} = \frac{2\sec^2(0)}{5\cos(0)} = \frac{2}{5}$$

↗

$\frac{0}{0}$-type

c. $$\lim_{x \to \infty} (5+x)^{1/x} = L \qquad \ln L = \lim_{x \to \infty} \ln(5+x)^{1/x}$$

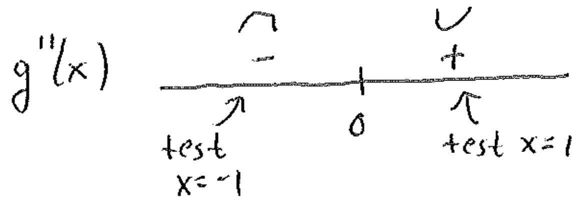
$$= \lim_{x \to \infty} \frac{1}{x} \ln(5+x) = \lim_{x \to \infty} \frac{\ln(5+x)}{x}$$

$$\text{L'Hopital} \to \quad = \lim_{x \to \infty} \frac{\frac{1}{5+x}}{1} = 0$$

$$L = e^0 = 1$$

6. Let $g(x) = 3x^5 + 20x^3$.

a. (6 points) Determine the open intervals where $g(x)$ is concave up and concave down. $g'(x) = 15x^4 + 60x^2$, $g''(x) = 60x^3 + 120x = 60x(x^2+2)$



concave up: $(0, \infty)$

concave down: $(-\infty, 0)$

b. (4 points) Determine all inflection points of $g(x)$. Just give the x-coordinates.

$x = 0$

7. (10 points) Use implicit differentiation to find the equation of the tangent line to the curve $x^3 + y^2 = 5y + 4$ at $(2, 1)$.

$3x^2 + 2y \cdot y' = 5y'$

$2yy' - 5y' = -3x^2$

$y'(2y-5) = -3x^2$

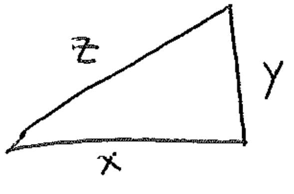
$$y' = \frac{-3x^2}{2y-5}$$

At $(2,1)$, $y' = \frac{-3 \cdot 4}{2-5} = 4$

Pt Slope: $y - y_1 = m(x - x_1)$

$\boxed{y - 1 = 4(x-2)}$

8. (10 points) Consider a right triangle with edges of length x, y, z, with z the hypotenuse. If x is increasing at a rate of 5 m/sec and z is increasing at a rate of 7 m/sec, at what rate is y increasing when $x = 3$ m and $z = 5$ m?



Given $dx/dt = 5$, $dz/dt = 7$

Find dy/dt when $x = 3$, $z = 5$

$z^2 = x^2 + y^2 \Rightarrow \cancel{2}z \frac{dz}{dt} = \cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt}$

$\Rightarrow y \frac{dy}{dt} = z \frac{dz}{dt} - x \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{y}\left(z \frac{dz}{dt} - x \frac{dx}{dt}\right)$

When $x = 3, z = 5$, $y^2 = 5^2 - 3^2 = 16$
$\Rightarrow y = 4$

$\frac{dy}{dt} = \frac{1}{4}(5 \cdot 7 - 3 \cdot 5) = \frac{20}{4} = \boxed{5 \text{ m/sec}}$

3. (8 points) Use the definition of derivative as a limit to find $f'(x)$ for $f(x) = 3x^2 - x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2 - (x+h) - (3x^2 - x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-x} - h \cancel{-3x^2} \cancel{+x}}{h} = \lim_{h\to 0} \frac{\cancel{h}(6x+3h-1)}{\cancel{h}}$$

$$= 6x - 1$$

4. (8 points each) Compute the following derivatives. **DO NOT SIMPLIFY**

a. $f'(t)$ where $f(t) = \cos^2(2t+1) = (\cos(2t+1))^2$

$$f'(t) = 2\cos(2t+1) \cdot (-)\sin(2t+1) \cdot 2$$

b. $\frac{d}{dx}\, x\ln(x^2+2) = x \cdot \frac{1}{x^2+2} \cdot 2x + \ln(x^2+2)$

c. $\frac{d}{dx}\,\frac{e^{5x}}{x^2+1} = \frac{(x^2+1)5e^{5x} - e^{5x} \cdot 2x}{(x^2+1)^2}$

5. (4 points each) Let $f(x) = x^2(x-4)^3$. Given: $f'(x) = x(x-4)^2(5x-8)$.

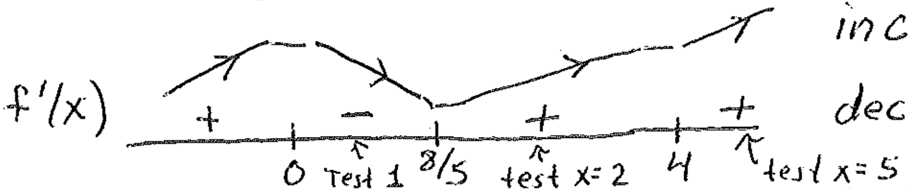
a. Find the critical points of $f(x)$. $f'(x) = 0$, $x = 0, 4, 8/5$

b. Find the open intervals where $f(x)$ is increasing and decreasing.

or $(\frac{8}{5}, 4) \cup (4, \infty)$

inc: $(-\infty, 0)$, $(\frac{8}{5}, \infty)$

dec: $(0, 8/5)$



c. Classify each critical point as a local minimum, local maximum or neither.

$x = 0$ is local max, $x = 4$ is neither

$x = 8/5$ is local min,

11. (8 points each) Evaluate the following integrals.

a. $\int \sin(\pi x/2) + 2^x - \frac{1}{\sqrt{1-x^2}}\,dx$

$$= -\frac{2}{\pi}\cos(\pi x/2) + \frac{2^x}{\ln(2)} - \sin^{-1}x + C$$

b. $\int \tan^3(2x)\ \sec^2(2x)\ dx$ $\qquad u = \tan(2x)$, $du = \sec^2(2x)\cdot 2\ dx$

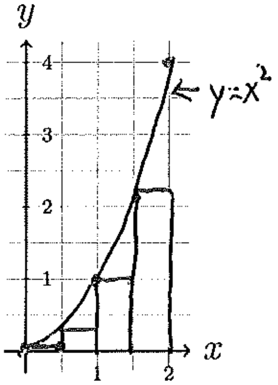
$$= \int u^3\,\frac{du}{2} = \frac{1}{2}\,\frac{u^4}{4} = \frac{1}{8}\tan^4(2x) + C$$

c. $\int_0^1 \frac{x+2}{x^2+4x+1}\,dx$ $\qquad u = x^2+4x+1$

$du = 2x+4\ dx = 2(x+2)\ dx$

$$= \int_1^6 \frac{du/2}{u} = \frac{1}{2}\ln|u|\Big|_1^6 = \frac{1}{2}\ln 6 \qquad (\text{since } \ln 1 = 0)$$

12. (8 points) Estimate the area below the curve $y = x^2$ over the interval [0,2] using L_4, the left end point approximation with four rectangles. Also, make a sketch of the graph of $y = x^2$ and illustrate the rectangles on your graph.



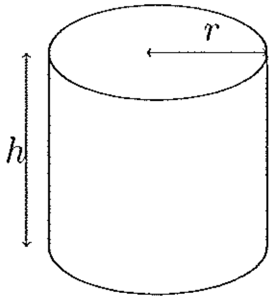
Divide [0,2] into 4 intervals of length $\frac{1}{2}$.

L_4 = Sum of areas of rectangles shown

$$= \frac{1}{2}\left(0 + \underbrace{\frac{1}{4} + 1 + \left(\frac{3}{2}\right)^2}_{\text{heights}}\right) = \frac{1}{2}\,\frac{14}{4} = \frac{7}{4}$$

$\nearrow$ base

9. (12 points) Find the dimensions of a cylinder with total surface area 6π square meters, including top and bottom, that maximizes its volume. (Recall, $V = \pi r^2 h$ and the side wall of the cylinder has area $2\pi r h$.)



Given $A = 6\pi$, maximize V.

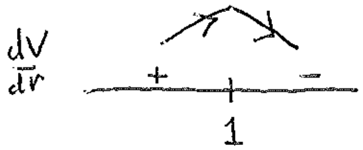
$$A = \underbrace{2\pi r h}_{\text{side}} + \underbrace{2\pi r^2}_{\text{top \& bottom}} = 6\pi$$

$$\Rightarrow rh + r^2 = 3 \Rightarrow rh = 3 - r^2, \; h = \frac{3-r^2}{r}$$

$$V = \pi r^2 h = \pi r^2 \frac{3-r^2}{r} = \pi r(3-r^2) = \pi(3r - r^3)$$

$$\frac{dV}{dr} = \pi(3-3r^2) = 0 \Rightarrow 3r^2 = 3 \Rightarrow r^2 = 1 \Rightarrow r = 1, \text{ since } r > 0.$$

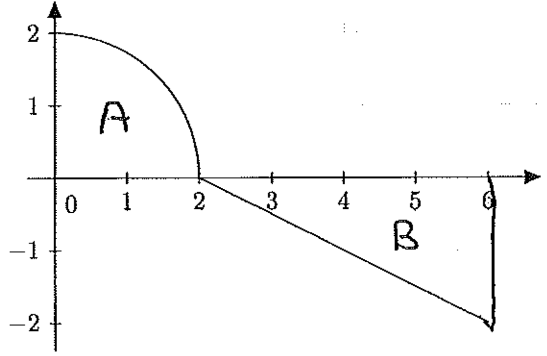
$$\text{when } r = 1, \; h = \frac{3-1^2}{1} = 2$$



← This is a max by first derivative test.

V is maximized when $r = 1\,m$, $h = 2\,m$

10. The velocity function $v = v(t)$ for an object moving along a straight line is graphed below. The horizontal axis is time measured in seconds, and the vertical axis is velocity in m/sec. The arc from (0,2) to (2,0) is a quarter circle.



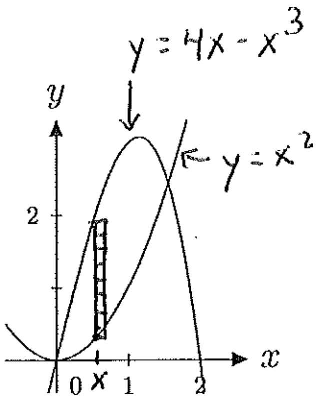
a. (5 points) Let $s = s(t)$ denote the position of the object. If the object is at position $s = 3$ when $t = 0$, where is it after 6 seconds?

$$s(6) - \underset{\overset{\nearrow}{s(0)=3}}{s(0)} = \int_0^6 v(t)\,dt = A - B = \tfrac{1}{4}\pi \cdot 2^2 - \tfrac{1}{2}\, 4 \cdot 2 = \pi - 4$$

$$s(6) = s(0) + \pi - 4 = \pi - 1 \; m$$

b. (5 points) Find the total distance the object travels during the time interval [0, 6] seconds. $A + B = \tfrac{1}{4}\pi \cdot 2^2 + \tfrac{1}{2}\, 4 \cdot 2 = \pi + 4 \;\; m$

16. Below is a sketch of the region bounded between the curves $y = 4x - x^3$ and $y = x^2$ for $x \geq 0$. Set up integrals for the following volumes but **do not evaluate the integrals.**



a. (4 points) Start by finding the point of intersection of the two curves with $x > 0$. Just give the x-coordinate.

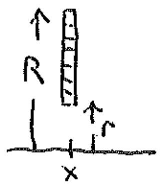
$4x - x^3 = x^2 \Rightarrow x^3 + x^2 - 4x = 0$, $x(x^2 + x - 4) = 0$

$x = 0$ $\quad$ $x = \frac{-1 \pm \sqrt{1^2 + 4 \cdot 4}}{2} = \frac{-1 \pm \sqrt{17}}{2}$

$x > 0 \Rightarrow x = \frac{-1 + \sqrt{17}}{2}$

b. (6 points) The volume of the solid obtained by rotating the region around the x-axis.

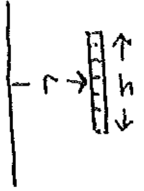
Washer: $dV = \pi(R^2 - r^2)\,dx = \pi\left((4x - x^3)^2 - (x^2)^2\right)\,dx$



$$V = \int_0^{\frac{-1+\sqrt{17}}{2}} \pi\left((4x - x^3)^2 - x^4\right)\,dx$$

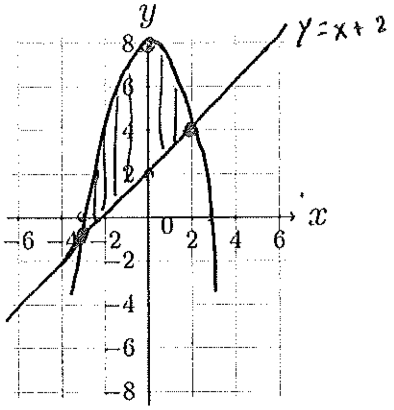
c. (6 points) The volume of the solid obtained by rotating the region around the y-axis.

$dV = 2\pi rh\,dx = 2\pi x\left(4x - x^3 - x^2\right)\,dx$



$$V = \int_0^{\frac{-1+\sqrt{17}}{2}} 2\pi x\left(4x - x^3 - x^2\right)\,dx$$

13. (12 points) Make a sketch of the region bounded between the parabola $y = 8 - x^2$ and the line $y = x + 2$, and then calculate its area.



Intersect: $8 - x^2 = x + 2$

$x^2 + x - 6 = 0$, $(x+3)(x-2) = 0$

$x = -3, y = -1$; $x = 2, y = 4$

$$A = \int_{-3}^{2} (8 - x^2) - (x+2)\, dx = \int_{-3}^{2} -x^2 - x + 6 \; dx$$

$$= -\frac{x^3}{3} - \frac{x^2}{2} + 6x \Big|_{-3}^{2} = \left(-\frac{8}{3} - 2 + 12\right) - \left(9 - \frac{9}{2} - 18\right)$$

$$= \frac{22}{3} - (-)\frac{27}{2} = \frac{22 \cdot 2 + 27 \cdot 3}{6} = \frac{125}{6}$$

44
81
125

14. (8 points) Solve the initial value problem: $f'(t) = \sqrt{t}$, $f(1) = 2$.

$$f(t) = \int \sqrt{t}\, dt = \int t^{1/2} dt = \frac{2}{3} t^{3/2} + C$$

$$f(1) = 2 \Rightarrow \quad 2 = \frac{2}{3} \cdot 1^{3/2} + C = \frac{2}{3} + C \Rightarrow C = 2 - \frac{2}{3} = 4/3$$

$$f(t) = \frac{2}{3} t^{3/2} + \frac{4}{3}$$

15. (8 points) a) Find the linear approximation of $f(x) = \sqrt{x}$ near $x = 4$.

$$f'(x) = \frac{1}{2} x^{-1/2}, \quad f'(4) = \frac{1}{2} 4^{-1/2} = \frac{1}{2} \frac{1}{\sqrt{4}} = \frac{1}{4}, \qquad f(4) = \sqrt{4} = 2$$

$$L(x) = f(a) + f'(a)(x - a)$$
$$= 2 + \frac{1}{4}(x - 4)$$

b) Use your estimate in part a) to estimate $\sqrt{4.1}$.

$$\sqrt{4.1} \approx L(4.1) = 2 + \frac{1}{4}(.1) = 2 + \frac{1}{4}\frac{1}{10} = 2 + \frac{1}{40}$$