Name Solutions Rec. Instr.

Signature Rec. Time

Math 220
Exam 3
April 7, 2016

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 75 minutes to complete the exam. Unless instructed otherwise, **show your work** on each problem.

Problem	Points	Points Possible	Problem	Points	Points Possible
1		16	6		12
2		10	7		4
3		9	8		10
4		10	9		5
5		12	10		12

Total Score: ________ out of 100

1. The function $f(x)$ and its first and second derivatives are:

$$f(x) = \frac{x^2-1}{x^2+3} \qquad f'(x) = \frac{8x}{(x^2+3)^2} \qquad f''(x) = \frac{-24(x^2-1)}{(x^2+3)^3}.$$

Find the information below about $f(x)$, and use it to sketch the graph of $f(x)$. When appropriate, write NONE. No work needs to be shown on this problem.

A. (1 point) Domain of $f(x)$: $(-\infty,\infty)$

B. (1 point) y-intercept: $-\frac{1}{3}$ $\quad f(0)=\frac{0^2-1}{0^2+3}=-\frac{1}{3}$

C. (1 point) x-intercept(s): ± 1 $\quad f(x)=\frac{(x-1)(x+1)}{x^2+3}=0 \iff x=\pm 1$

D. (1 point) Horizontal asymptote(s): $y=1$

$\lim_{x\to\infty} \frac{x^2-1}{x^2+3}=1 \qquad \lim_{x\to-\infty} \frac{x^2-1}{x^2+3}=1$

E. (1 point) Interval(s) $f(x)$ is increasing: $(0,\infty)$

F. (1 point) Interval(s) $f(x)$ is decreasing: $(-\infty,0)$

$f'(x)$ is always defined. $f'(x)=0 \iff x=0$

Sign of $f'(x)$ $\quad - \;\; \underset{0}{|} \;\; +$

G. (1 point) Local maximum(s) (x,y): None

H. (1 point) Local minimum(s) (x,y): $(0,-\frac{1}{3})$

I. (1 point) Interval(s) $f(x)$ is concave up: $(-1,1)$

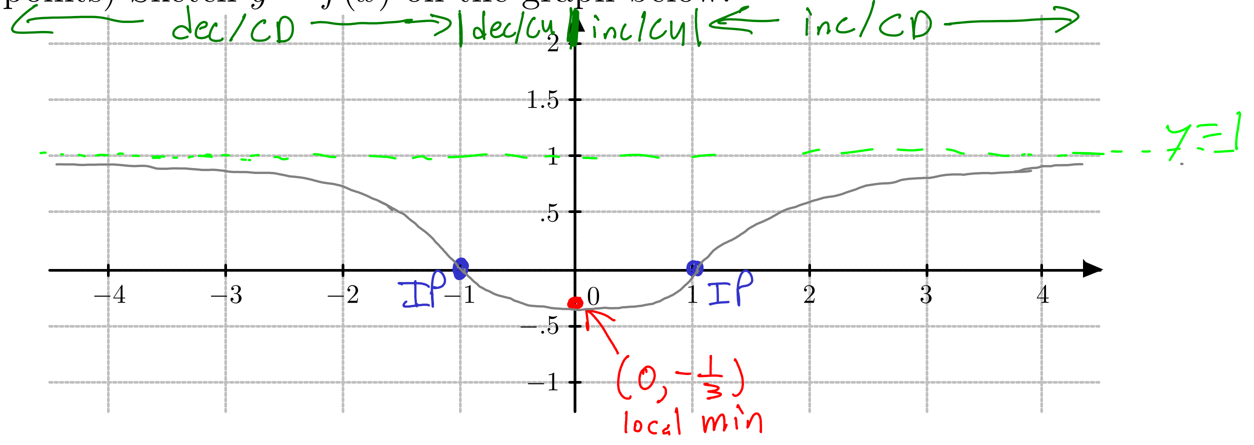
J. (1 point) Interval(s) $f(x)$ is concave down: $(-\infty,-1),(1,\infty)$

$f''(x)$ is always defined. $f''(x)=\frac{-24(x-1)(x+1)}{(x^2+3)^3}=0 \iff x=\pm 1$

Sign of $f''(x)$ $\quad - \;\; \underset{-1}{|} \;\; + \;\; \underset{1}{|} \;\; -$

K. (1 point) Inflection point(s) (x,y): $(-1,0),(1,0)$

L. (5 points) Sketch $y=f(x)$ on the graph below.



2. (10 points) Find the absolute maximum and absolute minimum of $g(x) = x^3 - 3x^2 + 4$ on $[-1, 1]$.

$g'(x) = 3x^2 - 6x = 3x(x-2)$ is defined everywhere

$g'(x) = 0 \iff x = 0$ or 2

$x = 0$ is the only critical number in $[-1, 1]$

$g(-1) = (-1)^3 - 3(-1)^2 + 4 = 0$

$g(0) = 0^3 - 3 \cdot 0^2 + 4 = 4$

$g(1) = 1^3 - 3 \cdot 1^2 + 4 = 2$

Absolute max at $(0, 4)$

Absolute min at $(-1, 0)$

3. A. (6 points) Find the linearization of $w(x) = \sqrt{x}$ at $x = 9$.

$w'(x) = \frac{1}{2\sqrt{x}}$. The linearization of $w(x)$ at $x=9$ is:

$L(x) = w(9) + w'(9)(x-9) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) = 3 + \frac{1}{6}(x-9)$

B. (3 points) Use your answer from Part **A** to estimate $\sqrt{9.6}$.

9.6 is close to 9 so

$\sqrt{9.6} = w(9.6) \approx L(9.6) = 3 + \frac{1}{6}(9.6-9) = 3 + \frac{1}{6}(.6) = 3.1$

4. (10 points) Find the function $v(x)$ satisfying $v''(x) = 2$, $v'(0) = -3$, and $v(0) = 5$.

$\int 2\,dx = 2x+C$ so $v'(x) = 2x+C$ for some constant C.

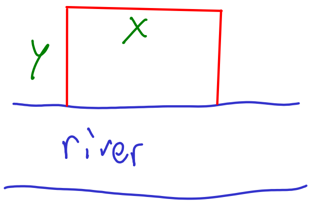
$-3 = v'(0) = 2\cdot 0 + C = C$ so $C = -3$. $v'(x) = 2x-3$.

$\int (2x-3)\,dx = x^2 - 3x + D$ so $v(x) = x^2 - 3x + D$ for some constant D.

$5 = v(0) = 0^2 - 3\cdot 0 + D = D$ so $D = 5$.

$v(x) = x^2 - 3x + 5$

5. (12 points) A farmer has 24 feet of fencing and wants to fence off a rectangular area that borders a straight river. The farmer needs no fencing along the river. What dimensions will maximize the fenced-in area? (Make sure to justify why your answer corresponds to the absolute maximum.)



Maximize Area: $A = xy$.

Length of Fencing: $x + 2y = 24$

$x = 24 - 2y$

$A(y) = (24-2y)\,y = 24y - 2y^2$.

$A'(y) = 24 - 4y$ is always defined.

$0 = A'(y) \iff 4y = 24 \iff y = 6$

Closed Interval Method

Maximize $A(y)$ on $[0,12]$

($y \geq 0$ and since $x \geq 0$, we must have $y \leq 12$)

$A(0) = (24 - 2\cdot 0)\cdot 0 = 0$

$A(6) = (24 - 2\cdot 6)\cdot 6 = 72$

$A(12) = (24 - 2\cdot 12)\cdot 12 = 0$

First Derivative

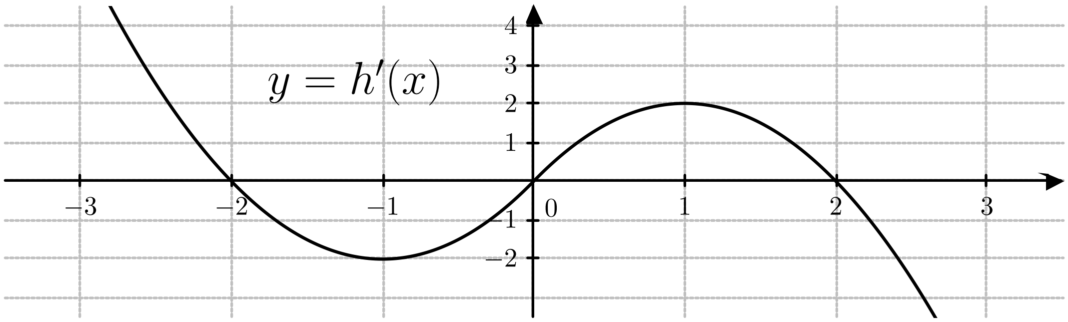
Sign of $A'(y)$ $\underset{}{+}\ \underset{6}{|}\ \underset{}{-}$

Second Derivative

$A''(y) = -4$

$A''(y) < 0$ for all y

Area is maximized by taking $y = 6$ ft and $x = 24 - 2\cdot 6 = 12$ ft.



6. (3 points each) $y = h'(x)$ is plotted above. Find:

A. Interval(s) where $h(x)$ is increasing: $(-\infty,-2), (0,2)$ decreasing: $(-2,0), (2,\infty)$

B. x-coordinate(s) where $h(x)$ has a local max: $-2, 2$ local min: 0

C. Interval(s) where $h(x)$ is concave up: $(-1,1)$ concave down: $(-\infty,-1), (1,\infty)$

D. x-coordinate(s) where $h(x)$ has an inflection point: $-1, 1$

7. (2 points each) In each of the following blanks, fill in "**max**" or "**min**".

A. If $l'(5) = 0$ and $l''(5) = 14$, then $l(x)$ has a local min at $x = 5$.

B. If $l'(2) = 0$ and $l''(2) = -3$, then $l(x)$ has a local max at $x = 2$.

8. (5 points each) Find the following most general antiderivatives. (I hope that you 'C' what I mean.)

A. $\int \left(\sec^2(x) + 4\right)\, dx =$ $\tan(x) + 4x + C$

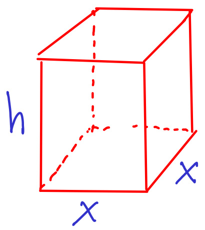
B. $\int \left(\sqrt{x} + 5e^x\right)\, dx =$ $\frac{2}{3} x^{3/2} + 5e^x + C$

9. (5 points) Find the differential dy if $y = \cos(4x^2)$.

$$\frac{dy}{dx} = -\sin(4x^2)\cdot 8x$$

$$dy = -\sin(4x^2)\cdot 8x \cdot dx$$

10. (12 points) A rectangular open-topped aquarium is to have a square base and volume 8 m^3. The material for the base costs \$2 per m^2, and the material for the sides costs \$1 per m^2. What dimensions minimize the cost of the aquarium? (Make sure to justify why your answer corresponds to an absolute minimum.)



Minimize Cost: $C = \underbrace{2\cdot x^2}_{\text{cost of base}} + \underbrace{1\cdot 4\cdot x h}_{\text{cost of the four sides}}$

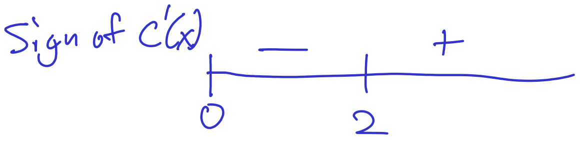
The volume is $x^2 h = 8$ so $h = \frac{8}{x^2}$.

Minimize $C(x) = 2x^2 + 4x\left(\frac{8}{x^2}\right) = 2x^2 + \frac{32}{x}$ on $(0,\infty)$

$C'(x) = 4x - \frac{32}{x^2}$ is defined on $(0,\infty)$.

$C'(x) = 0 \iff 4x - \frac{32}{x^2} = 0 \iff 4x = \frac{32}{x^2} \iff x^3 = 8 \iff x = 2$

First Derivative



Second Derivative

$C''(x) = 4 + \frac{64}{x^3} > 0$ on $(0,\infty)$

The cost of the aquarium is minimized by taking $x = 2$ m and $h = \frac{8}{2^2} = 2$ m.