Name Solutions Rec. Instr.

Signature Rec. Time

Math 220
Final Exam
May 11, 2016

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		18	8		6
2		5	9		4
3		10	10		3
4		3	11		6
5		12	12		6
6		6	13		7
7		7	14		7

Total Score:

1. (3 points each) Evaluate the following:

A. $\lim_{x \to \pi} \frac{\cos(x)}{x} =$ $\frac{\cos(\pi)}{\pi} = -\frac{1}{\pi}$

B. $\lim_{x \to \infty} \frac{23 + x - 5x^5 - 3x^9}{4x^9 - 3x - 2} =$ $-\frac{3}{4}$

C. $\frac{d}{dx} \int_0^x e^{\cos(t)}\, dt =$ $e^{\cos(x)}$

D. $\int \frac{7\, dx}{1 + x^2} =$ $7 \arctan(x) + C$

E. $\frac{d}{dx} \left(\frac{e^x}{\sqrt{x}} \right) =$ $\frac{e^x \cdot \sqrt{x} - e^x \cdot \left(\frac{1}{2\sqrt{x}}\right)}{x}$

F. $\frac{d}{dx} \left(\ln(x) \cdot \sin(x^2) \right) =$ $\frac{1}{x} \cdot \sin(x^2) + \ln(x) \cdot \cos(x^2) \cdot 2x$

2. (5 points) Using the **limit definition of the derivative**, find $f'(2)$ if $f(x) = x^2 + x$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\left((2+h)^2+(2+h)\right)-\left(2^2+2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+4h+h^2+\cancel{2}+h-\cancel{4}-\cancel{2}}{h} = \lim_{h\to 0} \frac{5h+h^2}{h}$$

$$= \lim_{h\to 0} (5+h) = 5+0 = 5$$

3. (5 points each) Find $\dfrac{dy}{dx}$ for:

A. $x^3 + y^3 = 5xy$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(5xy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 5y + 5x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 5x\frac{dy}{dx} = 5y - 3x^2$$

$$(3y^2-5x)\frac{dy}{dx} = 5y-3x^2$$

$$\frac{dy}{dx} = \frac{5y-3x^2}{3y^2-5x}$$

B. $y = x^{\cos(x)}$

$$\ln(y) = \ln\left(x^{\cos(x)}\right) = \cos(x)\cdot\ln(x)$$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left[\cos(x)\cdot\ln(x)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\sin(x)\cdot\ln(x) + \cos(x)\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(-\sin(x)\cdot\ln(x) + \frac{\cos(x)}{x}\right) = x^{\cos(x)}\left(-\sin(x)\cdot\ln(x) + \frac{\cos(x)}{x}\right)$$

4. (1 point each) For the function $w(x)$, one has $w''(x) = \dfrac{2(x-1)}{x^2+3}$. Find the following:

A. Interval(s) where $w(x)$ is concave up: $(1, \infty)$

B. Interval(s) where $w(x)$ is concave down: $(-\infty, 1)$

C. x-coordinate(s) where $w(x)$ has an inflection point: $x = 1$

5. (6 points each) Evaluate the following:

A. $\displaystyle\int t\sqrt{t^2+3}\,dt$ $= \displaystyle\int \sqrt{u}\,\frac{du}{2} = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(t^2+3)^{3/2} + C$

$u = t^2+3$

$du = 2t\,dt$

$\dfrac{du}{2} = t\,dt$

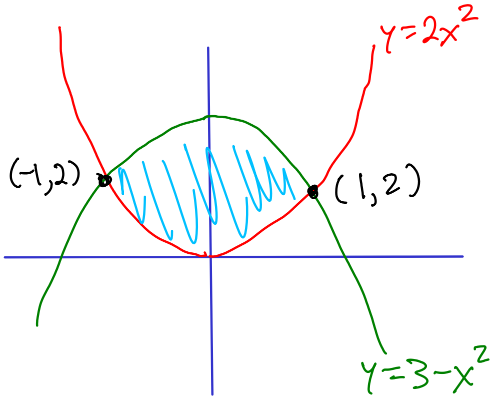
B. $\displaystyle\int_0^{\pi/2} \sin^3(\theta)\cos(\theta)\,d\theta$ $= \displaystyle\int_0^1 u^3\,du = \frac{u^4}{4}\Big|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$

$u = \sin(\theta)$

$du = \cos(\theta)\,d\theta$

θ	u
$\frac{\pi}{2}$	1
0	0

6. (6 points) Find the area bounded between $y = 2x^2$ and $y = 3 - x^2$.



$2x^2 = 3 - x^2 \iff 3x^2 = 3 \iff x^2 = 1 \iff x = \pm 1$

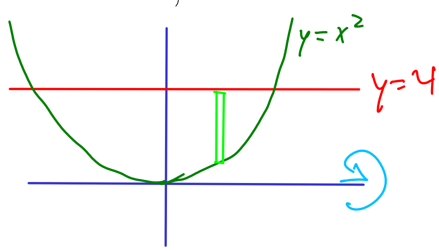
so the curves intersect at $x = \pm 1$.

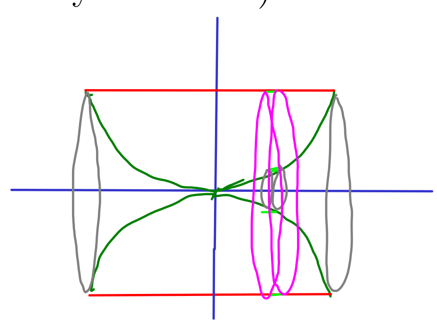
$$\text{AREA} = \int_{-1}^{1} \left((3-x^2) - 2x^2\right) dx = \int_{-1}^{1} (3 - 3x^2)\, dx$$

$$= 3x - x^3 \Big|_{-1}^{1} = (3\cdot 1 - 1^3) - (3\cdot(-1) - (-1)^3)$$

$$= 2 - (-2) = 4$$

7. (7 points) Find the volume of the solid obtained by rotating the region bounded by $y = 4$ and $y = x^2$ around the x-axis. You do not need to simplify the numeric expression in your final answer. (Your final answer should only include numbers and π; it should not include any variables.)



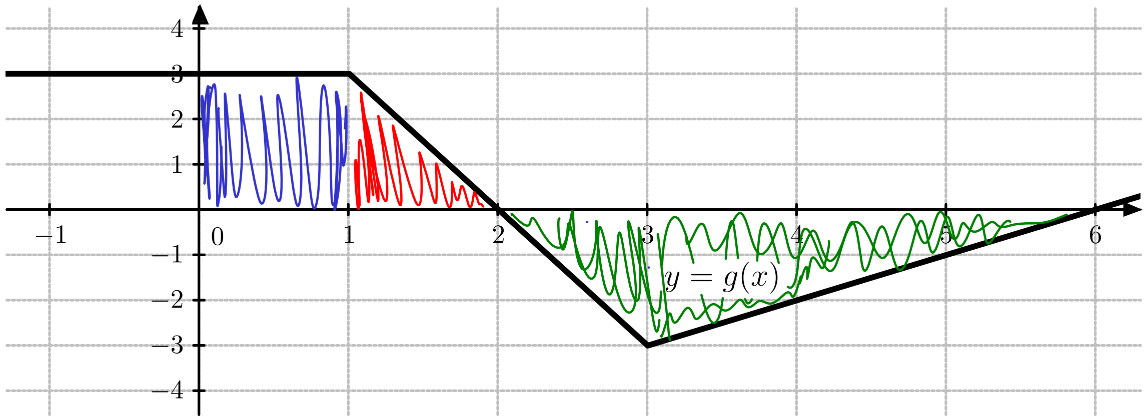


$x^2 = 4$ when $x = \pm 2$ so the curves intersect at $x = \pm 2$.

$$\text{Volume} = \int_{-2}^{2} \pi 4^2\, dx - \int_{-2}^{2} \pi (x^2)^2\, dx = \int_{-2}^{2} (16\pi - \pi x^4)\, dx$$

$$= 16\pi x - \frac{\pi}{5} x^5 \Big|_{-2}^{2} = \left(16\pi\cdot 2 - \frac{\pi}{5} 2^5\right) - \left(16\pi(-2) - \frac{\pi}{5}(-2)^5\right)$$

$$= 32\pi - \frac{32\pi}{5} + 32\pi - \frac{32\pi}{5} = \frac{256\pi}{5}$$

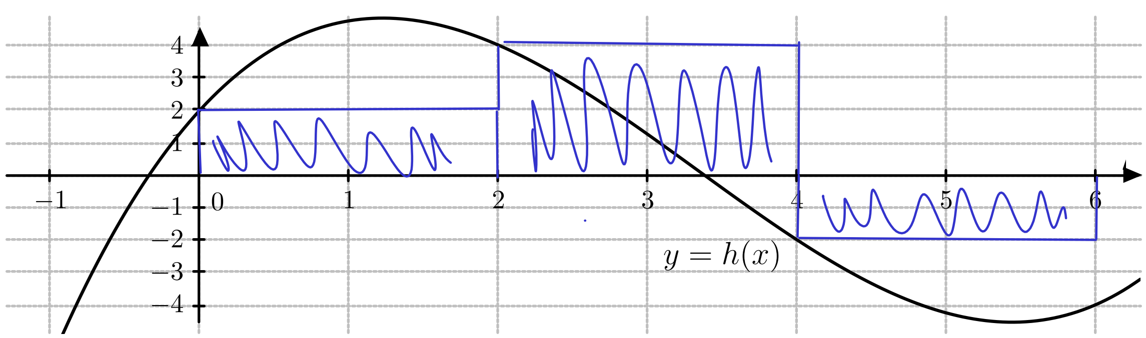


8. (2 points each) $y = g(x)$ is plotted above. Evaluate the following definite integrals. (No work needs to be shown.)

A. $\int_0^1 g(x)\,dx =$ $1 \cdot 3 = 3$

B. $\int_2^1 g(x)\,dx =$ $-\int_1^2 g(x)\,dx = -\frac{1}{2}\cdot 1 \cdot 3 = -\frac{3}{2}$

C. $\int_2^6 g(x)\,dx =$ $-\frac{1}{2}\cdot 4 \cdot 3 = -6$



9. (4 points) $y = h(x)$ is plotted above. Estimate $\int_0^6 h(x)\,dx$ by using a Riemann sum with $n = 3$ subintervals, taking the sampling points to be left endpoints (the Left Hand Rule L_3). Also, illustrate the rectangles on the graph above.

$\Delta x = \frac{6-0}{3} = 2$

$\int_0^6 h(x)\,dx \approx f(0)\cdot \Delta x + f(2)\cdot \Delta x + f(4)\cdot \Delta x$

$= 2\cdot 2 + 4 \cdot 2 - 2\cdot 2 = 8$

10. (3 points) Let $r(t)$ denote the rate in gallons/minute that water flows into a boat t minutes after a crash. Describe the meaning of $\int_0^5 r(t)\,dt$.

By the Net Change Theorem, the integral represents the volume of water in gallons that entered the boat during the five minutes directly after the crash

11. (6 points) Use a linearization to approximate sin(.01).

Let $f(x) = \sin(x)$. $\quad f'(x) = \cos(x)$.

The linearization of $f(x)$ near $x=0$ is

$$L(x) = f(0) + f'(0)(x-0) = \sin(0) + \cos(0)(x-0) = x.$$

$$\sin(.01) = f(.01) \underset{\uparrow}{\approx} L(.01) = .01$$

.01 is close to 0

12. (6 points) Find the absolute minimum and maximum of $v(x) = x^3 - 3x + 1$ on the interval $[0, 2]$.

$v'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$ is defined everywhere.

$v'(x) = 0$ when $x = \pm 1$. The only critical number in $[0,2]$ is $x=1$.

$v(0) = 0^3 - 3\cdot 0 + 1 = 1$

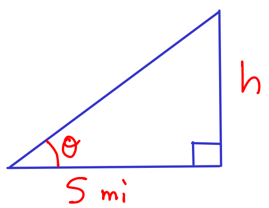
$v(1) = 1^3 - 3\cdot 1 + 1 = -1$

$v(2) = 2^3 - 3\cdot 2 + 1 = 3$

On $[0,2]$, $v(x)$ has an absolute minimum at $(1,-1)$ and an absolute maximum at $(2,3)$.

13. (7 points) A hot air balloon rising vertically is tracked by an observer located 5 miles from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{4}$, and it is changing at a rate of $\frac{1}{10}$ radians/minute. How fast is the balloon rising at this moment?

Want: $\frac{dh}{dt}$ when $\theta = \frac{\pi}{4}$ rad, $\frac{d\theta}{dt} = \frac{1}{10}\ \frac{\text{rad}}{\text{min}}$



$$\tan(\theta) = \frac{h}{5}$$

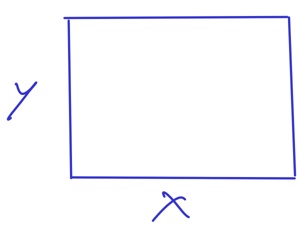
$$\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\frac{h}{5}$$

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{5}\frac{dh}{dt}$$

$$\frac{dh}{dt} = 5\sec^2(\theta)\frac{d\theta}{dt} = \frac{5\cdot\frac{d\theta}{dt}}{\cos^2(\theta)}$$

When $\theta = \frac{\pi}{4}$ and $\frac{d\theta}{dt} = \frac{1}{10}$, $\frac{dh}{dt} = \frac{5\cdot\frac{1}{10}}{\cos^2(\frac{\pi}{4})} = \frac{\frac{1}{2}}{(\frac{1}{\sqrt{2}})^2} = 1$ mi/min

14. (7 points) Suppose that you want to enclose a 25 ft^2 rectangular area with fencing. What is the minimum length of fencing needed? (Make sure to justify why your answer corresponds to the absolute minimum.)



We want to minimize the perimeter $p = 2x + 2y$.

The area is $xy = 25$ ft^2 so $y = \frac{25}{x}$.

$p(x) = 2x + 2\left(\frac{25}{x}\right) = 2x + \frac{50}{x}$.

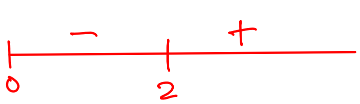
We want to minimize $p(x)$ on $(0,\infty)$.

$p'(x) = 2 - \frac{50}{x^2}$ is defined on $(0,\infty)$. $p'(x) = 0$ when $2 = \frac{50}{x^2} \Leftrightarrow x^2 = 25 \Leftrightarrow x = \pm 5$

The only critical number in $(0,\infty)$ is $x = 5$.

First Derivative Justification

Sign of $p'(x)$



Second Derivative Justification

$p''(x) = \frac{100}{x^3} > 0$ on $(0,\infty)$

The perimeter is minimized by taking $x = 5$ ft and $y = \frac{25}{5} = 5$ ft, making the perimeter $2\cdot 5 + 2\cdot 5 = 20$ ft.