Name Solutions Rec. Instr.

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Math 220

Final Exam

December 13, 2017

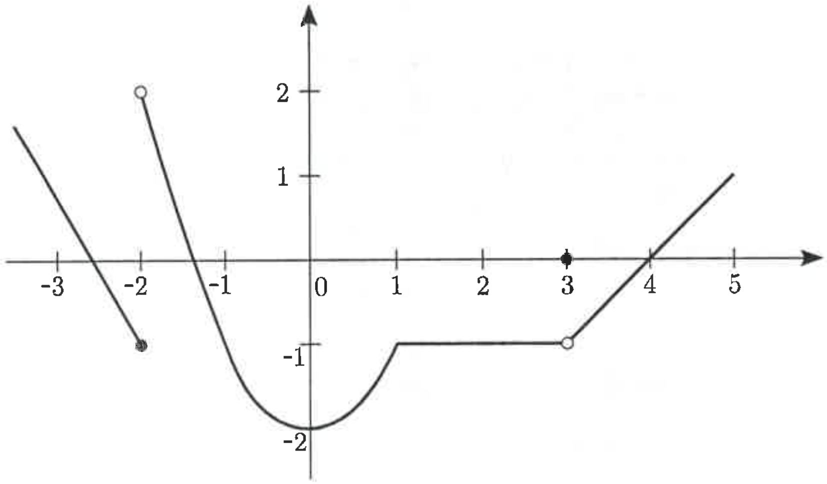
No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 1 hour and 50 minutes to complete the exam.

Total = 200 points. **Show your work unless stated otherwise.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		12	9		10
2		18	10		10
3		8	11		8
4		24	12		24
5		12	13		12
6		10	14		8
7		10	15		8
8		10	16		16

Total Score:

1. (2 points each) Evaluate the following for the function $f(x)$ graphed below or state that it does not exist. No work needs to be shown.



a. $\lim_{x \to -2^+} f(x) =$ 2

$\nwarrow x > -2$

b. $\lim_{x \to 1} f(x) =$ -1

c. $\lim_{x \to 3} f(x) =$ -1

d. $f'(3.5) =$ 1 (slope of tangent line)

e. Indicate all values of x between -3 and 4 at which $f(x)$ is not continuous.

$-2, 3$

f. Indicate all values of x between -3 and 4 at which $f'(x)$ is not defined.

$-2, 1, 3$

2. (6 points each) Evaluate the following limits.

a. $\lim_{x \to 2} \dfrac{x^3 - 4x}{2 - x} = \lim_{x \to 2} \dfrac{x(x^2-4)}{2-x} = \lim_{x \to 2} \dfrac{x\cancel{(x-2)}(x+2)}{-1\cancel{(2-x)}}$

$= \lim_{x \to 2} -x(x+2) = -2 \cdot 4 = -8$

b. $\lim_{h \to 0} \dfrac{\sin(3h)}{\sin(2h)} + \dfrac{\cos(3h)}{\cos(2h)} = \lim_{h \to 0} \dfrac{\sin(3h)}{\sin(2h)} + \lim_{h \to 0} \dfrac{\cos(3h)}{\cos(2h)}$

$= \lim_{h \to 0} \dfrac{3\cos(3h)}{2\cos(2h)} + \dfrac{\cos(0)}{\cos(0)}$

L'Hop.

$= \dfrac{3}{2} \dfrac{1}{1} + \dfrac{1}{1} = \dfrac{3}{2} + 1 = \dfrac{5}{2}$

c. $\lim_{x\to\infty} (x^2+5)^{1/x} = L$

$$\ln L = \lim_{x\to\infty} \ln(x^2+5)^{1/x} = \lim_{x\to\infty} \frac{1}{x}\ln(x^2+5)$$

$$= \lim_{x\to\infty} \frac{\ln(x^2+5)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^2+5}\cdot 2x}{1}$$

$\frac{\infty}{\infty}$-type, L'Hopital

$$= \lim_{x\to\infty} \frac{2x}{x^2+5} = \lim_{x\to\infty} \frac{2}{2x} = 0$$

$\frac{\infty}{\infty}$ L'Hop

Thus $\boxed{L = e^0 = 1}$

3. (8 points) Use the definition of derivative as a limit to find $f'(x)$ for $f(x) = x^2 + x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+(x+h)-(x^2+x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+h+1)}{\cancel{h}}$$

$$= 2x+1.$$

4. (8 points each) Compute the following derivatives. **DO NOT SIMPLIFY**

a. $f'(t)$ where $f(t) = \sin^5(\ln t)$ $= (\sin(\ln t))^5$

$$f'(t) = 5(\sin(\ln t))^4 \cdot \cos(\ln(t)) \cdot \frac{1}{t}$$, using chain-rule

b. $\frac{d}{dx} e^{3x} \tan^{-1}(x)$ (Here, $\tan^{-1}(x) = \arctan(x)$.)

$$= e^{3x} \cdot \frac{1}{1+x^2} + \tan^{-1}(x) \cdot e^{3x} \cdot 3$$

product rule

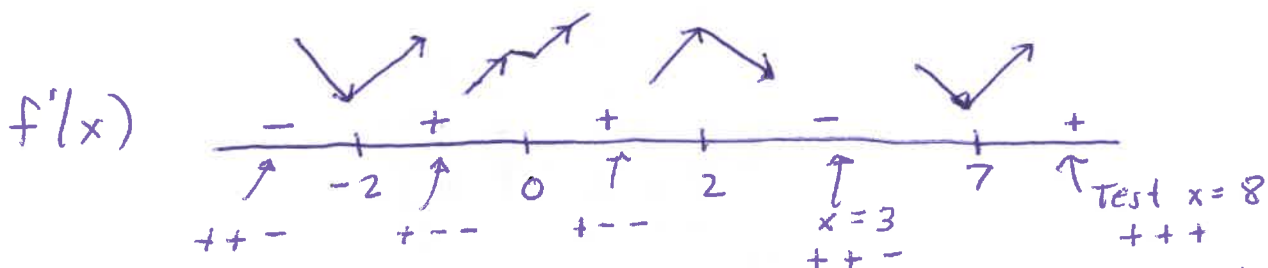
c. $\frac{d}{dx} \frac{x + \tan x}{1 - x^2} = \frac{(1-x^2)(1+\sec^2 x) - (x + \tan x)(-2x)}{(1-x^2)^2}$

quotient rule

5. (4 points each) Let $f(x)$ be a function with $f'(x) = x^2(x^2 - 4)(x - 7)$.

a. Find the critical points of $f(x)$. $x = 0, 2, -2, 7$

b. Find the open intervals where $f(x)$ is increasing and decreasing.



increasing: $(-2,0), (0,2), (7,\infty)$ (or $(-2,2), (7,\infty)$)

decreasing: $(-\infty,-2), (2,7)$

c. Classify each critical point as a local minimum, local maximum or neither.

-2 local min

0 neither

2 local max

7 local min

6. Let $g(x) = 2x^6 - 5x^4$.

a. (6 points) Determine the open intervals where $g(x)$ is concave up and concave down.

$g'(x) = 12x^5 - 20x^3$

$g''(x) = 60x^4 - 60x^2 = 60x^2(x^2 - 1) = 60x^2(x-1)(x+1)$

$g''(x)$: $\cup$ $+$ | -1 | $\cap$ $-$ | 0 | $\cap$ $-$ | 1 | $\cup$ $+$

$+--$ $+-+$ $x = 1/2$: $+-+$ Test $x=2$: $+++$

concave up: $(-\infty,-1), (1,\infty)$

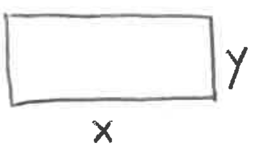
concave down: $(-1,0), (0,1)$

or $(-1,1)$

b. (4 points) Determine all inflection points of $g(x)$. Just give the x-coordinates.

$x = -1, 1$ (where concavity changes)

7. (10 points) Consider a rectangle with edges of length x, y. If x is increasing at a rate of 5 m/sec and y is decreasing at a rate of 2 m/sec, at what rate is the area A of the rectangle changing when $x = 3$ m and $y = 4$ m?



$A = xy$

Given $dx/dt = 5 \frac{m}{sec}$, $\frac{dy}{dt} = -2 \frac{m}{sec}$,

Find dA/dt when $x = 3, y = 4$.

$\frac{dA}{dt} = \frac{d}{dt}(xy) = x\frac{dy}{dt} + y\frac{dx}{dt}$, by product rule

$= x(-2) + y \cdot 5$) substitute $x=3, y=4$

$= 3(-2) + 4 \cdot 5$

$= -6 + 20$

$= \boxed{14 \ m^2/sec}$

8. (10 points) Use implicit differentiation to find the equation of the tangent line to the curve $xy + y^2 = 2x - 1$ at $(2, 1)$.

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(2x-1)$$

$$\Rightarrow xy' + y + 2yy' = 2$$

$$\Rightarrow y'(x+2y) = 2 - y$$

$$\Rightarrow y' = \frac{2-y}{x+2y}$$

$$\Rightarrow y'\Big|_{(2,1)} = \frac{2-1}{2+2} = \frac{1}{4}$$

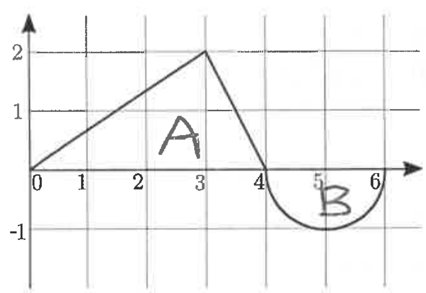
Tangent line:

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \frac{1}{4}(x-2)$$

or $y = \frac{1}{4}x + \frac{1}{2}$

9. The velocity function $v = v(t)$ for an object moving along a straight line is graphed below. The horizontal axis is time measured in seconds, and the vertical axis is velocity in m/sec. The arc from (4,0) to (6,0) is a semicircle.



a. (2 points) State the time intervals when the object is moving to the right, and when it is moving to the left.

Right $(v>0)$: $(0,4)$, Left $(v<0)$: $(4,6)$

b. (4 points) Let $s = s(t)$ denote the position of the object. If the object is at position $s = -3$ when $t = 0$, where is it after 6 seconds?

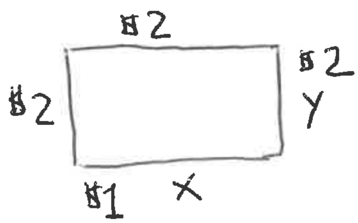
$$s(6) - s(0) = \int_0^6 v\,dt = A - B = \frac{1}{2}4 \cdot 2 - \frac{1}{2}\pi 1^2 = 4 - \frac{1}{2}\pi$$

$$s(6) = -3 + 4 - \frac{1}{2}\pi = 1 - \frac{1}{2}\pi$$

c. (4 points) Find the total distance the object travels during the time interval $[0, 6]$ seconds.

Total distance $= \int_0^6 |v|\,dt = A + B = 4 + \frac{1}{2}\pi$

10. (10 points) A rectangular fence consists of three sides costing \$2 per meter, and one side costing \$1 per meter. If the area of the rectangle is 12 square meters, find the dimensions that minimize the cost of the fence.



$C = \text{cost} = \underset{\text{bottom}}{x} + \underset{\text{top}}{2x} + \underset{\text{left}}{2y} + \underset{\text{right}}{2y} = 3x + 4y$

$A = xy = 12 \Rightarrow y = 12/x$

Given that $A = 12$, minimize C.

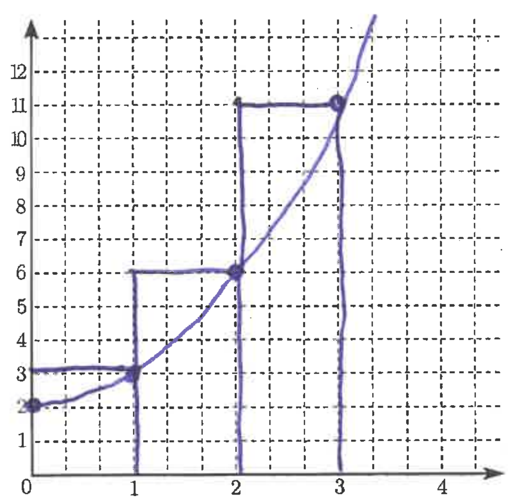
$C = 3x + 4y = 3x + 4 \cdot \frac{12}{x} = 3x + 48x^{-1}$

$\frac{dC}{dx} = 3 - 48x^{-2} = 0 \Rightarrow 3 = \frac{48}{x^2} \Rightarrow x^2 = 16 \Rightarrow x = 4m$ $(x \geq 0)$

$y = \frac{12}{4} = 3.m$

This is a minimum since $C \to \infty$ as $x \to \infty$ and as $x \to 0^+$.

11. (8 points) Estimate the area below the curve $y = x^2 + 2$ over the interval [0,3] using R_3, the right end point approximation with three rectangles. Also, make a sketch of the graph of $y = x^2 + 2$ and illustrate the rectangles on your graph.



$R_3 = 3 \cdot 1 + 6 \cdot 1 + 11 \cdot 1$

$= 20$

12. (8 points each) Evaluate the following integrals.

a. $\displaystyle\int e^{5x} - \frac{1}{\sqrt{4-x^2}}\,dx$ $= \displaystyle\int e^{5x}\,dx - \int \frac{dx}{\sqrt{4-x^2}}$ ↙ let $x = 2u$, $dx = 2\,du$

$= \frac{1}{5}e^{5x} - \displaystyle\int \frac{2\,du}{\sqrt{4-4u^2}} = \frac{1}{5}e^{5x} - \frac{2}{2}\int \frac{du}{\sqrt{1-u^2}}$

$= \frac{1}{5}e^{5x} - \sin^{-1}(u) + C$, $u = x/2$

$= \frac{1}{5}e^{5x} - \sin^{-1}(x/2) + C$

b. $\displaystyle\int \sin^5(2x)\cos(2x)\,dx$

$u = \sin(2x)$

$du = \cos(2x)\cdot 2\,dx$

$= \displaystyle\int u^5 \frac{du}{2} = \frac{1}{2}\frac{u^6}{6} + C$

$= \frac{1}{12}\sin^6(2x) + C$

c. $\displaystyle\int_1^e \frac{(\ln x)^2}{x}\,dx$

$u = \ln x$

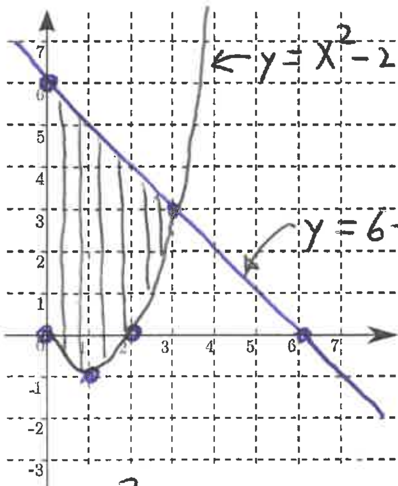
$du = \frac{1}{x}\,dx$

$x=1 \Rightarrow u = \ln(1) = 0$

$x = e \Rightarrow u = \ln(e) = 1$

$= \displaystyle\int_0^1 u^2\,du = \left.\frac{u^3}{3}\right|_0^1 = \boxed{\frac{1}{3}}$

13. (12 points) Make a sketch of the region with $x \geq 0$ bounded by the y-axis, the parabola $y = x^2 - 2x$ and the line $y = 6 - x$, and then calculate its area.



Parabola: $y = x^2 - 2x = x(x-2)$
intercepts $x = 0, 2$, vertex: $x = 1$, $y = -1$

Intersection: $x^2 - 2x = 6 - x$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3, y = 3$; $x = -2$, not relevant

$$A = \int_0^3 (6-x) - (x^2 - 2x)\, dx = \int_0^3 6 + x - x^2\, dx = 6x + \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^3$$

$$= 18 + \frac{9}{2} - 9 = \frac{27}{2} \text{ or } 13.5$$

14. (8 points) Solve the initial value problem: $f'(t) = 4t^3 - \sin t$, $f(0) = 1$.

$f(t) = \int 4t^3 - \sin(t)\, dt = t^4 + \cos(t) + C$

$f(0) = 1 \Rightarrow 1 = 0 + \cos(0) + C = 1 + C \Rightarrow C = 0$

$f(t) = t^4 + \cos(t)$

15. (8 points) a) Find the linear approximation of $f(x) = \sqrt{x}$ near $x = 9$.

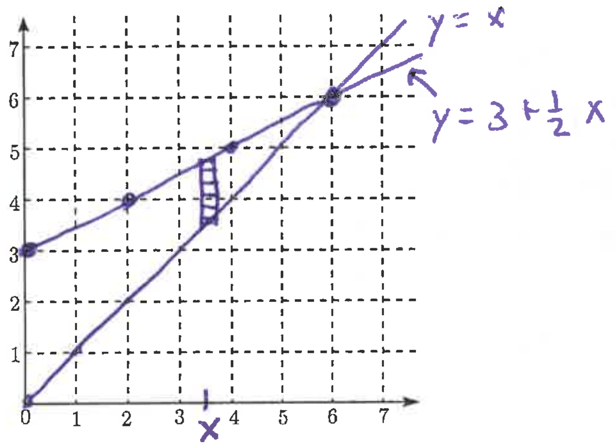
$L(x) = f(9) + f'(9)(x-9)$ $\qquad f(9) = \sqrt{9} = 3$, $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$, $f'(9) = \frac{1}{6}$

$= 3 + \frac{1}{6}(x-9)$

b) Use your estimate in part a) to estimate $\sqrt{8.9}$.

$\sqrt{8.9} \approx L(8.9) = 3 + \frac{1}{6}(8.9 - 9) = 3 + \frac{1}{6}\frac{-1}{10} = 3 - \frac{1}{60}$

16. a. (4 points) Sketch the region bounded by the y-axis and the lines $y = x$, $y = 3 + \frac{1}{2}x$, and find the indicated volumes.



Intersection $x = 3 + \frac{1}{2}x$

$\Rightarrow \frac{1}{2}x = 3 \Rightarrow x = 6$

$y = 6$

b. (6 points) The volume of the solid obtained by rotating the region around the y-axis. Just set up the integral. You do not need to evaluate it.

r $\quad h$

$$dV = 2\pi r h\, dx = 2\pi x\left(3 + \tfrac{1}{2}x - x\right) dx$$
$$= 2\pi x\left(3 - \tfrac{1}{2}x\right) dx$$

$$V = 2\pi \int_0^6 x\left(3 - \tfrac{1}{2}x\right) dx$$

c. (6 points) The volume of the solid obtained by rotating the region around the x-axis. Just set up the integral. You do not need to evaluate it.

R $\quad r$

$$dV = \pi\left(R^2 - r^2\right) dx$$
$$= \pi\left(\left(3 + \tfrac{1}{2}x\right)^2 - x^2\right) dx$$

$$V = \pi \int_0^6 \left(3 + \tfrac{1}{2}x\right)^2 - x^2 \, dx$$