Name Solutions Rec. Instr.
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Math 220
Exam 3
April 6, 2017

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 75 minutes to complete the exam. Unless instructed otherwise, **show your work** on each problem.

Problem	Points	Points Possible	Problem	Points	Points Possible
1		16	6		10
2		10	7		10
3		12	8		6
4		6	9		12
5		18	Total Score		100

1. The function $f(x)$ and its first and second derivatives are:

$$f(x) = \frac{x^2}{x^2+3} \qquad f'(x) = \frac{6x}{(x^2+3)^2} \qquad f''(x) = \frac{-18(x^2-1)}{(x^2+3)^3}.$$

Find the information below about $f(x)$, and use it to sketch the graph of $f(x)$. When appropriate, write NONE. No work needs to be shown on this problem.

A. (1 point) Domain of $f(x)$: $(-\infty, \infty)$

B. (1 point) y-intercept: $f(0) = 0$ so $(0,0)$

C. (1 point) x-intercept(s): $\frac{x^2}{x^2+3} = 0$ when $x = 0$ so $(0,0)$

D. (1 point) Horizontal asymptote(s): $y = 1$

$\lim_{x \to \infty} \frac{x^2}{x^2+3} = 1$ $\qquad \lim_{x \to -\infty} \frac{x^2}{x^2+3} = 1.$

E. (1 point) Interval(s) $f(x)$ is increasing: $(0, \infty)$

F. (1 point) Interval(s) $f(x)$ is decreasing: $(-\infty, 0)$

$f'(x)$ is always defined.

$f'(x) = 0$ when $x = 0$.

$f(x)$: ↘ ↗

sign of $f'(x)$: $-$ | $+$

0

G. (1 point) Local maximum(s) (x, y): none

H. (1 point) Local minimum(s) (x, y): $(0,0)$

I. (1 point) Interval(s) $f(x)$ is concave up: $(-1, 1)$

J. (1 point) Interval(s) $f(x)$ is concave down: $(-\infty, -1)$ and $(1, \infty)$

$f''(x)$ is always defined.

$f''(x) = 0$ when $x = \pm 1$

$f(x)$: CD CU CD

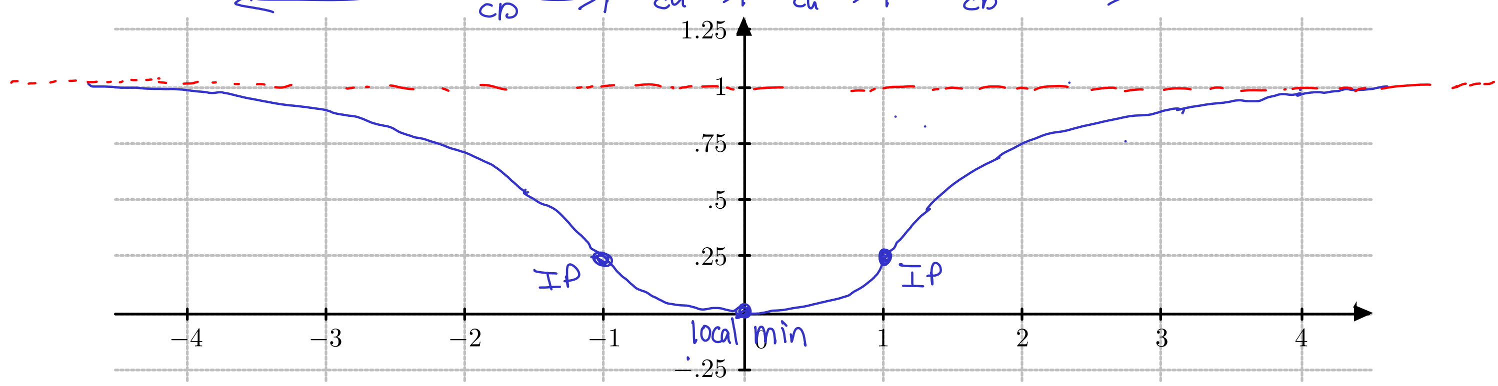
sign of $f''(x)$: $-$ | $+$ | $-$

-1 $\qquad 1$

K. (1 point) Inflection point(s) (x, y): $(-1, \frac{1}{4})$ and $(1, \frac{1}{4})$

L. (5 points) Sketch $y = f(x)$ on the graph below.

⟵ dec CD ⟶ | ⟵ dec CU ⟶ | ⟵ inc CU ⟶ | ⟵ inc CD ⟶



2. (10 points) Find the absolute maximum and absolute minimum of $w(x) = 2x^3 - 9x^2 + 3$ on $[-1, 1]$.

$w'(x) = 6x^2 - 18x = 6x(x-3)$ is always defined.
$w'(x) = 0$ when $x = 0$ and $x = 3$. the only critical number in $[-1, 1]$ is at $x = 0$.

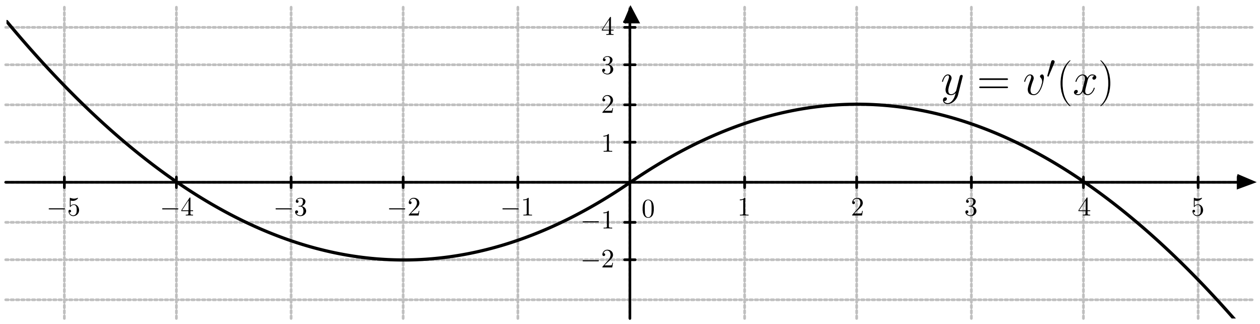
$w(-1) = 2\cdot(-1)^3 - 9\cdot(-1)^2 + 3 = -8$

$w(0) = 2\cdot 0^3 - 9\cdot 0^2 + 3 = 3$

$w(1) = 2\cdot 1^3 - 9\cdot 1^2 + 3 = -4$

Absolute max: $(0, 3)$

Absolute min: $(-1, -8)$



3. (3 points each) $y = v'(x)$ is plotted above. Find:

A. Interval(s) where $v(x)$ is increasing: $(-\infty, -4), (0, 4)$ decreasing: $(-4, 0), (4, \infty)$

B. x-coordinate(s) where $v(x)$ has a local max: $x = -4, 4$ local min: $x = 0$

C. Interval(s) where $v(x)$ is concave up: $(-2, 2)$ concave down: $(-\infty, -2), (2, \infty)$

D. x-coordinate(s) where $v(x)$ has an inflection point: $x = -2, 2$

4. (3 points each) In each of the following blanks, fill in "**max**" or "**min**".

A. If $h'(3) = 0$ and $h''(3) = -52$, then $h(x)$ has a local __max__ at $x = 3$.

B. If $h'(-2) = 0$ and $h''(-2) = 37$, then $h(x)$ has a local __min__ at $x = -2$.

5. (6 points each) Find the following limits. (Use limit notation correctly.)

A. $\lim_{x\to\infty} \frac{e^x + 5x}{x+3} \underset{LH}{=} \lim_{x\to\infty} \frac{e^x+5}{1} = \infty$

(type $\frac{\infty}{\infty}$)

B. $\lim_{\theta\to 0} \frac{\sin(\theta^2)}{3\theta^2} \underset{LH}{=} \lim_{\theta\to 0} \frac{\cos(\theta^2)\cdot 2\theta}{6\theta} = \lim_{\theta\to 0} \frac{\cos(\theta^2)}{3} = \frac{\cos(0^2)}{3} = \frac{1}{3}$

type $\frac{0}{0}$

C. $\lim_{x\to -\infty} \frac{3x+5}{\sqrt{4x^2+7x}} = \lim_{x\to -\infty} \frac{\frac{3x+5}{x}}{\frac{\sqrt{4x^2+7x}}{x}} = \lim_{x\to-\infty} \frac{3+\frac{5}{x}}{\frac{\sqrt{4x^2+7x}}{-\sqrt{x^2}}}$

$\boxed{x<0 \text{ so } x = -\sqrt{x^2}}$

$= \lim_{x\to-\infty} \frac{3+\frac{5}{x}}{-\sqrt{\frac{4x^2+7x}{x^2}}} = \lim_{x\to-\infty} \frac{3+\frac{5}{x}}{-\sqrt{4+\frac{7}{x}}}$

$= \frac{3+0}{-\sqrt{4+0}} = -\frac{3}{2}$

6. (10 points) Let $p(x) = 100 - 2x$ be the price in dollars per cake a bakery can charge if it sells x cakes. What cake price will maximize revenue? (Recall, revenue is the total amount of money received from the sale of x cakes. Make sure to justify why your answer corresponds to an absolute maximum.)

The revenue generated from the sale of x cakes is

$R(x) = x \cdot p(x) = x \cdot (100 - 2x) = 100x - 2x^2.$

$R'(x) = 100 - 4x$ is always defined. $R'(x) = 0$ when $x = \frac{100}{4} = 25$ cakes.

<u>1st Derivative Justification</u>

	$x<25$	$x>25$
$R(x)$	$\nearrow$	$\searrow$
sign of $R'(x)$	$+$	$-$

$R(x)$ is increasing for $x < 25$.
$R(x)$ is decreasing for $x > 25$.

<u>2nd Derivative Justification</u>

$R''(x) = -4 < 0$ for all x.

$R(x)$ is always concave down.

Revenue is maximized when $x = 25$ cakes are sold at a price of $p(25) = 100 - 2 \cdot 25 = 50$ \$/cake.

7. A. (7 points) Find the linearization of $g(x) = \ln(x)$ at $x = 1$.

$g'(x) = \frac{1}{x}$. The linearization of $g(x)$ at $x = 1$ is

$L(x) = g(1) + g'(1)(x-1) = \ln(1) + \frac{1}{1}(x-1) = x - 1.$

B. (3 points) Use your answer from Part **A** to estimate $\ln(1.15)$.

$$\ln(1.15) = g(1.15) \approx L(1.15) = 1.15 - 1 = .15$$

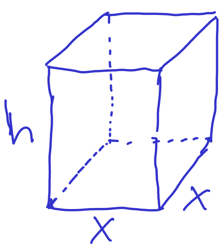
$\boxed{1.15 \text{ is close to } 1}$

8. (6 points) The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. Find the differential dV.

$$\frac{dV}{dr} = 4\pi r^2$$

$$dV = 4\pi r^2 \cdot dr$$

9. (12 points) A rectangular open-topped aquarium is to have a square base and volume 5 m^3. The material for the base costs \$10 per m^2, and the material for the sides costs \$1 per m^2. What dimensions minimize the cost of the aquarium? (Make sure to justify why your answer corresponds to an absolute minimum.)



Minimize Cost: $C = \underbrace{10x^2}_{\text{(cost of base)}} + \underbrace{1 \cdot 4xh}_{\text{(cost of four sides)}}$

We are given that the volume $x^2 h = 5$ m^3 so $h = \frac{5}{x^2}$.

Minimize $C(x) = 10x^2 + 4x\left(\frac{5}{x^2}\right) = 10x^2 + \frac{20}{x}$ on $(0, \infty)$.

$C'(x) = 20x - \frac{20}{x^2}$ is always defined on $(0,\infty)$. $C'(x) = 0$ when

$20x - \frac{20}{x^2} = 0 \iff 20x = \frac{20}{x^2} \iff x^3 = 1 \iff x = 1$ m.

The only critical number in $(0,\infty)$ is at $x = 1$m

<u>1st Derivative Justification</u>

	$(0,1)$	$(1,\infty)$
$C(x)$	$\searrow$	$\nearrow$
sign of $C'(x)$	$-$	$+$

$C(x)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$

<u>2nd Derivative Justification</u>

$C''(x) = 20 + \frac{40}{x^3} > 0$ on $(0,\infty)$

$C(x)$ is concave up on $(0,\infty)$

The cost is minimized when $x = 1$m and $h = \frac{5}{1^2} = 5$m.

$\boxed{1\text{m by } 1\text{m by } 5\text{m}}$