Name Solutions Rec. Instr.
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Math 220
Final Exam
May 10, 2017

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		18	8		5
2		5	9		7
3		10	10		3
4		3	11		6
5		12	12		6
6		6	13		6
7		7	14		6

Total Score:

1. (3 points each) Evaluate the following:

A. $\lim\limits_{x\to 0} \dfrac{e^x}{x+1} = \dfrac{e^0}{0+1} = 1$

B. $\lim\limits_{\theta\to 0} \dfrac{\sin(\theta^2)}{\theta} = \lim\limits_{\theta\to 0} \dfrac{\cos(\theta^2)\cdot 2\theta}{1} = \dfrac{\cos(0^2)\cdot 2\cdot 0}{1} = 0$

$\left(\text{type } \frac{0}{0}\right)$

C. $\displaystyle\int \left(\sqrt{x} + \cos(x) - 5\right)\, dx = \frac{2}{3} x^{3/2} + \sin(x) - 5x + C$

D. $\dfrac{d}{dx} \displaystyle\int_3^x t \cdot \sin(t^3)\, dt = x \cdot \sin(x^3)$

E. $\dfrac{d}{dx}\left(\dfrac{\tan(x)}{e^x+5}\right) = \dfrac{\sec^2(x)\cdot(e^x+5) - \tan(x)\cdot e^x}{(e^x+5)^2}$

F. $\dfrac{d}{dx}\left(\cos(x^3)\cdot \arctan(x)\right) = -\sin(x^3)\cdot 3x^2 \cdot \arctan(x) + \cos(x^3)\cdot \dfrac{1}{1+x^2}$

2. (5 points) Using the **limit definition of the derivative**, find $f'(2)$ if $f(x) = 3x^2$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3(2+h)^2 - 3\cdot 2^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{12} + 12h + 3h^2 - \cancel{12}}{h} = \lim_{h\to 0} \frac{12h + 3h^2}{h}$$

$$= \lim_{h\to 0} (12 + 3h) = 12 + 3\cdot 0 = 12$$

3. (5 points each) Find $\frac{dy}{dx}$ for:

A. $x^2 - xy + y^3 = 5$

$$\frac{d}{dx}(x^2 - x\cdot y + y^3) = \frac{d}{dx} 5$$

$$2x - y - x\cdot\frac{dy}{dx} + 3y^2\cdot\frac{dy}{dx} = 0$$

$$2x - y = x\cdot\frac{dy}{dx} - 3y^2\frac{dy}{dx} = (x - 3y^2)\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 3y^2}$$

B. $y = x^{7x}$

$$\ln(y) = \ln(x^{7x}) = 7x\cdot\ln(x)$$

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}(7x\cdot\ln(x))$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 7\cdot\ln(x) + 7x\cdot\frac{1}{x} = 7\cdot\ln(x) + 7$$

$$\frac{dy}{dx} = y\cdot(7\cdot\ln(x) + 7) = x^{7x}\cdot(7\cdot\ln(x) + 7)$$

4. (1 point each) For the function $w(x)$, one has $w''(x) = \dfrac{3-x}{x^2+7}$. Find the following:

A. Interval(s) where $w(x)$ is concave up: $(-\infty, 3)$

B. Interval(s) where $w(x)$ is concave down: $(3, \infty)$

C. x-coordinate(s) where $w(x)$ has an inflection point: $x=3$

5. (6 points each) Evaluate the following:

A. $\displaystyle\int \frac{\sqrt{\ln(x)}}{x}\,dx = \int \sqrt{u} \cdot du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\ln(x))^{3/2} + C$

$u = \ln(x)$

$\dfrac{du}{dx} = \dfrac{1}{x}$

$du = \dfrac{1}{x} \cdot dx$

B. $\displaystyle\int_0^{\pi/2} \sin^4(\theta)\cos(\theta)\,d\theta = \int_0^1 u^4\,du = \frac{u^5}{5}\Big|_0^1 = \frac{1^5}{5} - \frac{0^5}{5} = \frac{1}{5}$

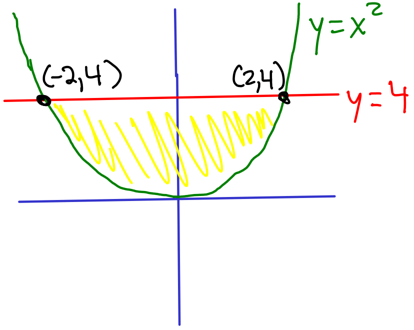
$u = \sin(\theta)$

$\dfrac{du}{d\theta} = \cos(\theta)$

$du = \cos(\theta)\,d\theta$

θ	u
$\frac{\pi}{2}$	1
0	0

6. (6 points) Find the area bounded between $y = 4$ and $y = x^2$.



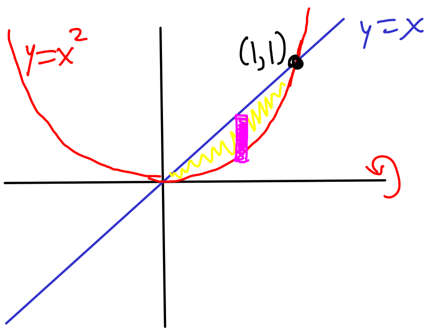
$$\text{AREA} = \int_{-2}^{2} |4-x^2|\,dx = \int_{-2}^{2} (4-x^2)\,dx$$

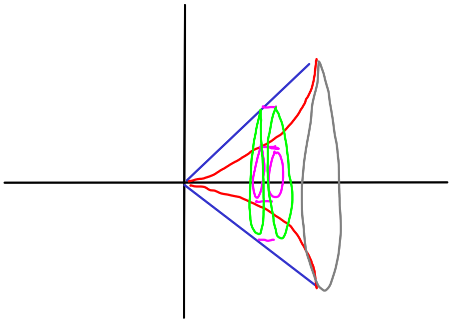
$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2}$$

$$= \left(4\cdot 2 - \frac{2^3}{3}\right) - \left(4\cdot(-2) - \frac{(-2)^3}{3}\right)$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = \frac{32}{3}$$

7. (7 points) Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ around the x-axis.



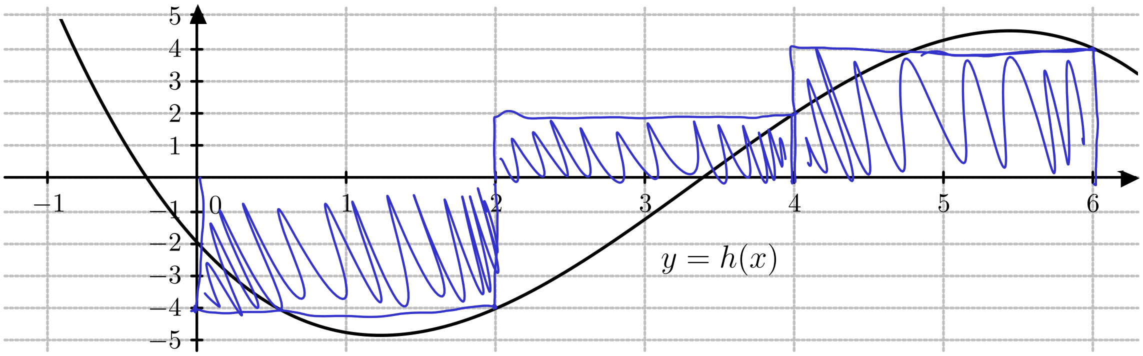


$$\text{Volume} = \overset{\text{outer volume}}{\int_0^1 \pi\cdot x^2\,dx} - \overset{\text{inner volume}}{\int_0^1 \pi(x^2)^2\,dx}$$

$$= \frac{\pi x^3}{3}\Big|_0^1 - \frac{\pi x^5}{5}\Big|_0^1$$

$$= \left(\frac{\pi\cdot 1^3}{3} - \frac{\pi\cdot 0^3}{3}\right) - \left(\frac{\pi\cdot 1^5}{5} - \frac{\pi\cdot 0^5}{5}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}$$

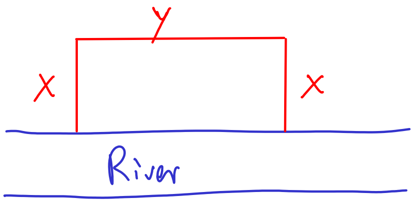


8. (5 points) $y = h(x)$ is plotted above. Estimate $\int_0^6 h(x)\,dx$ by using a Riemann sum with $n = 3$ subintervals, taking the sampling points to be right endpoints (the Right Hand Rule R_3). Also, illustrate the rectangles on the graph above.

$$\Delta x = \frac{6-0}{3} = 2$$

$$\int_0^6 h(x)\,dx \approx R_3 = h(2)\cdot\Delta x + h(4)\cdot\Delta x + h(6)\cdot\Delta x$$
$$= (-4)\cdot 2 + 2\cdot 2 + 4\cdot 2$$
$$= 4$$

9. (7 points) A farmer has 20 feet of fencing and wants to fence off a rectangular area that borders a straight river. The farmer needs no fencing along the river. What dimensions will maximize the fenced-in area? (Make sure to justify why your answer corresponds to the absolute maximum.)



We want to maximize the area $A = xy$. We know $2x + y = 20$ ft so $y = 20 - 2x$.

Maximize $A(x) = x(20-2x) = 20x - 2x^2$ where $0\text{ ft} \le x \le 10\text{ ft}$.

$A'(x) = 20 - 4x$ is defined everywhere, and $A'(x) = 0$ when $0 = 20 - 4x \Leftrightarrow 4x = 20 \Leftrightarrow x = 5$ ft. The only critical number for $A(x)$ is $x = 5$.

1st Derivative Justification $A(x)$ ↗ ↘ sign of $A'(x)$ + | − 5	2nd Derivative Justification $A''(x) = -4 < 0$ for all x $A(x)$ is always concave down	$A(0) = 20\cdot 0 - 2\cdot 0^2 = 0$ ft^2 $A(5) = 20\cdot 5 - 2\cdot 5^2 = 50$ ft^2 $A(10) = 20\cdot 10 - 2\cdot 10^2 = 0$ ft^2

The area is maximized when $x = 5$ ft and $y = 20 - 2\cdot 5 = 10$ ft.

10. (3 points) Let $r(t)$ be the rate of growth of a cat in pounds per week, where t denotes the number of weeks since the cat's birth. What does $\int_2^5 r(t)\,dt$ represent?

Let $W(t)$ denote the weight of the cat in pounds t weeks after birth. $W'(t)=r(t)$. By the Net Change Theorem, $\int_2^5 r(t)dt = \int_2^5 W'(t)dt = W(5)-W(2)$, which is the amount of weight in pounds gained by the cat from age 2 weeks to age 5 weeks.

11. (6 points) Use a linearization of $u(x) = \sqrt{x}$ at $x = 9$ to approximate $\sqrt{9.6}$.

$u'(x)=\frac{1}{2\sqrt{x}}$. The linearization of $u(x)$ at $x=9$ is

$$L(x)=u(9)+u'(9)(x-9)=\sqrt{9}+\frac{1}{2\cdot\sqrt{9}}\cdot(x-9)=3+\frac{1}{6}(x-9)$$

$$\sqrt{9.6}=u(9.6)\underset{\overset{\uparrow}{\text{9.6 is close to 9}}}{\approx} L(9.6)=3+\frac{1}{6}(9.6-9)=3+\frac{1}{6}\cdot(.6)=3.1$$

12. (6 points) Find the absolute minimum and maximum of $v(x) = x^3 + 3x^2 + 1$ on the interval $[-1, 1]$.

$v'(x)=3x^2+6x=3x(x+2)$ is always defined.

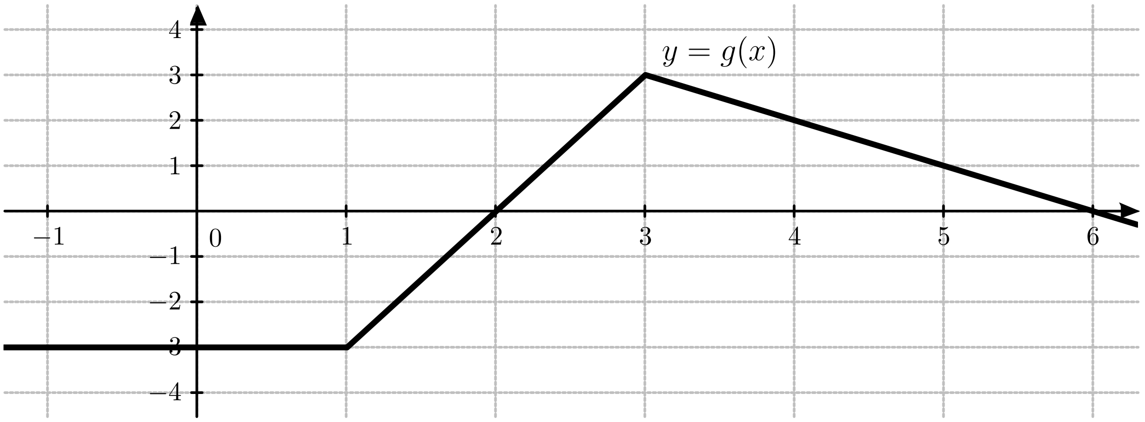
$v'(x)=0$ when $x=0$ or $x=-2$. The only critical number of $v(x)$ in $[-1,1]$ is at $x=0$.

$$v(-1)=(-1)^3+3\cdot(-1)^2+1=3$$

$$v(0)=0^3+3\cdot 0^2+1=1$$

$$v(1)=1^3+3\cdot 1^2+1=5$$

On the interval $[-1,1]$, $v(x)$ has an absolute min at $(0,1)$ and an absolute max at $(1,5)$.



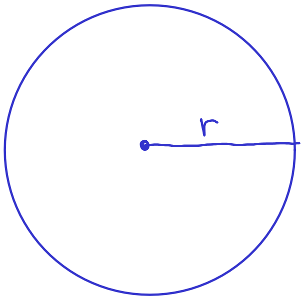
13. (2 points each) $y = g(x)$ is plotted above. Evaluate the following definite integrals.

A. $\int_0^1 g(x)\,dx = -3$

B. $\int_2^1 g(x)\,dx = -\int_1^2 g(x)\,dx = -\left(-\frac{1}{2}\cdot 1\cdot 3\right) = \frac{3}{2}$

C. $\int_2^6 g(x)\,dx = \frac{1}{2}\cdot 4\cdot 3 = 6$

14. (6 points) Suppose that an oil spill from a ruptured tanker spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 ft/sec, how fast is the area of the spill increasing when the radius is 10 ft?



Area $A = \pi r^2$

Want: $\frac{dA}{dt}$ when $r = 10$ ft

Know: $\frac{dr}{dt} = 2$ ft/s

$A = \pi r^2$

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = 2\pi r\cdot\frac{dr}{dt}$$

When $r = 10$ ft,

$$\frac{dA}{dt} = 2\pi\cdot 10\cdot 2 = 40\pi \text{ ft}^2/\text{s}$$