EXAM 1: MATH 220 - Calculus 1

SOLUTIONS

June 21st 2017

Name:

Instructor:

1	2	3	4	5	6	7	8	Total
20	6	8	6	12	16	20	12	

Instructions: You have 1 hour and 15 minutes to complete this exam. Show all of your work. Calculators are not allowed.

1) (4 points each) Calculate the following limits:

a) $\lim_{u\to 4} \frac{\sqrt{u}-2}{4-u} = \lim_{u\to 4} \frac{\sqrt{u}-2}{4-u} \frac{(\sqrt{u}+2)}{(\sqrt{u}+2)} = \lim_{u\to 4} \frac{u-4}{(4-u)(\sqrt{u}+2)}$

$= \lim_{u\to 4} \frac{-\cancel{(4-u)}^{1}}{\cancel{(4-u)}(\sqrt{u}+2)} = \lim_{u\to 4} \frac{-1}{\sqrt{u}+2} = -\frac{1}{4}$

b) $\lim_{x\to 2}(\underbrace{3x^2+5x+x^{1/2}-6}_{\text{continuous at } x=2}) = 3(2)^2+5(2)+2^{1/2}-6$

$= 16+\sqrt{2}$

c) $\lim_{\theta\to 0} \frac{cos(\theta)-cos^2(\theta)}{4\theta} = \lim_{\theta\to 0} \frac{\cos\theta(1-\cos\theta)}{4\theta}$

$= \left(\underbrace{\lim_{\theta\to 0} \frac{\cos\theta}{4}}_{1}\right)\left(\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}\right) = 1\cdot 0 = 0$

d) $\lim_{x\to 1} \frac{2x^2-x-1}{x-1} = \lim_{x\to 1} \frac{(2x+1)\cancel{(x-1)}^{1}}{\cancel{(x-1)}} = \lim_{x\to 1}(2x+1) = 3$

e) $\lim_{x\to 0} \frac{(x+1)sin(x)}{x} = \left(\lim_{x\to 0}(x+1)\right)\left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) = 1\cdot 1 = 1$

2) (6 points) Find the limits at infinity for the following functions (Make sure to consider both $+\infty$ and $-\infty$, if necessary.)

a) $f(x) = \frac{6x^3+x^2-3}{-2x^3-3x^2+x-1}$

$$\lim_{x\to\pm\infty} f(x) = -\frac{6}{2} = -3$$

b) $g(x) = \frac{4x^3+x+7}{x-2}$

$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to+\infty} \frac{4x^3}{x} = 4\lim_{x\to+\infty} x^2 = \infty$$

3) (8 points) Show that the equation $2e^t = t^4$ has a solution on the interval $[-1, 0]$. (Hint: Use the Intermediate Value Theorem)

Define $f(t) = 2e^t - t^4$

we want to evaluate f at the endpts of our interval:

$f(-1) = 2e^{-1} - (-1)^4 = \frac{2}{e} - 1 < 0$

$f(0) = 2e^0 - 0 = 2 > 0$

Now, since $f(t)$ is continuous on $[-1, 0]$ due to continuity of $2e^t$ and t^4 for all real numbers, then we can conclude by the IVT that there is a value "c" in $(0,1)$ such that $f(c) = 0$

So it must be true that $2e^c = c^4$.

4) (6 points) Find $\lim_{x \to 0}(5x^2 \cos(\frac{2}{x}))$ using the Squeeze Theorem.

We know that

$$-1 \le \cos\left(\frac{2}{x}\right) \le 1$$

Multiply by $5x^2$, and we get

$$-5x^2 \le 5x^2 \cos\left(\frac{2}{x}\right) \le 5x^2$$

Take $\lim_{x \to 0}(-5x^2) \le \lim_{x \to 0}\left(5x^2 \cos\left(\frac{2}{x}\right)\right) \le \lim_{x \to 0}(5x^2)$

$$0 \le \underbrace{\lim_{x \to 0}\left(5x^2 \cos\left(\frac{2}{x}\right)\right)}_{\substack{\| \\ 0}} \le 0$$

by Squeeze Theorem

5) (12 points) Let $f(x) = 2x^2 - x$. Compute f'(-1) using the limit definition, and find an equation to the tangent line at a=-1.

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{(2x^2 - x) - (2(-1)^2 - (-1))}{x+1}$$

$$= \lim_{x \to -1} \frac{2x^2 - x - 3}{x+1} = \lim_{x \to -1} \frac{(2x-3)\cancel{(x+1)}^{1}}{\cancel{(x+1)}} = \lim_{x \to -1}(2x-3) = \boxed{-5}$$

OR $$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0} \frac{2(-1+h)^2 - (-1+h) - 3}{h}$$

$$= \lim_{h \to 0} \frac{2(1+h^2-2h)+1-h-3}{h} = \lim_{h \to 0} \frac{\cancel{2} + 2h^2 - 4h + \cancel{1} - h - \cancel{3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2h-5)}{\cancel{h}} = \lim_{h \to 0}(2h-5) = \boxed{-5}$$

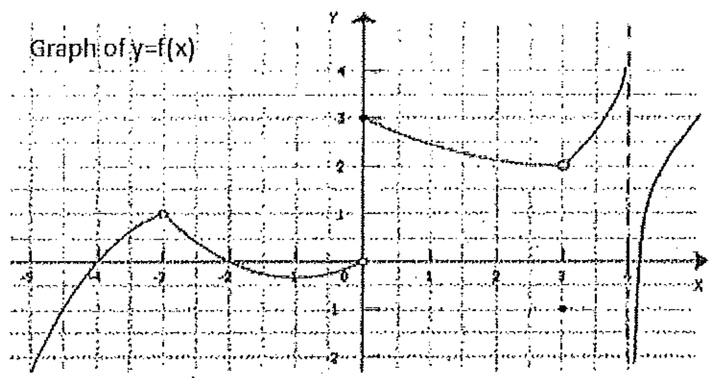
<u>Equation of tangent line</u>

$$y - f(-1) = f'(-1)(x - (-1))$$

$$y - 3 = -5(x+1)$$

$\star$ Can combine to have

$$g(x) = \begin{cases} f(x), & \text{if } x \neq 3, -3 \\ 1, & \text{if } x = -3 \\ 2, & \text{if } x = 3 \end{cases}$$



6) (16 points) For each point of discontinuity of the graph of the function y=f(x) above, provide the following information:
a) Type of discontinuity
b) The right and left-hand limits.
c) For any removable discontinuity, how would you redefine f(x) so that it is continuous at those points?
d) For any jump disconuity, is f(x) right- or left-continuous at those points?

- $x=-3$, removable discontinuity

  $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = 1$

  We could redefine our function as $g(x) = \begin{cases} f(x), & x \neq -3 \\ 1, & x = -3 \end{cases}$ $\star$

- $x=0$, jump discontinuity

  $\lim_{x \to 0^-} f(x) = 0$, $\underbrace{\lim_{x \to 0^+} f(x) = 3 = f(3)}$

  This means f(x) is right-continuous at x=0

- $x=3$, removable discontinuity

  $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 2$

  We can redefine our function as $g(x) = \begin{cases} f(x), & x \neq 3 \\ 2, & x = 3 \end{cases}$ $\star$

- $x=4$, infinite discontinuity

  $\lim_{x \to 4^+} f(x) = -\infty$, $\lim_{x \to 4^-} f(x) = \infty$

7) (4 points each) Calculate the following:

a) Find $\frac{dy}{dx}$ for $y = 7x^4 + 3x^2 - 2x + 1$.

$$\frac{dy}{dx} = 28x^3 + 6x - 2$$

b) $f(z) = \frac{3z^2+1}{e^z+2}$. Find $f'(z)$.

$$f'(z) = \frac{(e^z+2)(6z) - (3z^2+1)(e^z)}{(e^z+2)^2}$$

c) $g(x) = (2x^{2/3} - 3x^2)(x^{-1/2} + 5e^x)$

$$g'(x) = (2x^{2/3} - 3x^2)(-\tfrac{1}{2}x^{-3/2} + 5e^x) + (\tfrac{4}{3}x^{-1/3} - 6x)(x^{-1/2} + 5e^x)$$

d) $\frac{d}{dx}(\frac{1}{4}x^2 - x^{1/2})$ at x=9

$$\frac{d}{dx}\left(\tfrac{1}{4}x^2 - x^{1/2}\right)\Big|_{x=9} = \left(\tfrac{1}{2}x - \tfrac{1}{2}x^{-1/2}\right)\Big|_{x=9}$$

$$= \frac{9}{2} - \frac{1}{2}\cdot\frac{1}{\sqrt{9}} = \frac{9}{2} - \frac{1}{6} = \frac{26}{6} = \frac{13}{2}$$

e) Let $y = x^6 - 3x^4 + x^3 - 3x + 2$. Find y".

$$y' = 6x^5 - 12x^3 + 3x^2 - 3$$

$$y'' = 30x^4 - 36x^2 + 6x$$

8) (12 points) Let $s(t) = 50t - \frac{2}{3}t^3$ be a function giving the height of a balloon in feet at time t seconds, $0 \leq t \leq 8.6$.

a) State/write out the equation for how would you find the velocity function v(t) using the limit definition. Then find v(t) using any method.

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

$$v(t) = 50 - 2t^2$$

b) Find the time t which gives v(t)=0. What is the height of the balloon at this time? (Note: you do not need to simplify your answer).

$$0 = 50 - 2t^2$$
$$0 = 2(25 - t^2)$$
$$0 = 25 - t^2$$

So $t = 5$ (can't be -5 since we are on $0 \leq t \leq 8.6$)

Then $s(5) = 50(5) - \frac{2}{3}(5)^3$
$= 250 - \frac{250}{3} = \frac{500}{3}$

* Note: this is the maximum height of the balloon