SOLUTIONS

# EXAM 2: MATH 220 - Calculus 1

July 11th 2017

Name:

Instructor:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

Instructions: You have 1 hour and 15 minutes to complete this exam. Show all of your work. Calculators are not allowed.

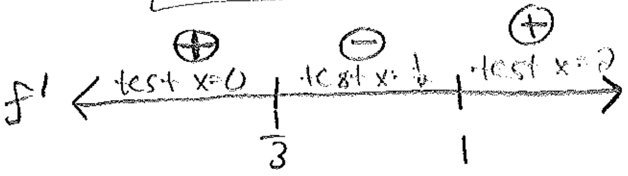
1) (16 points) Let $f(x) = x^3 - 2x^2 + x - 1$. Find the intervals on which f is concave up or concave down, increasing or decreasing, the points of inflection, the critical points, and the local minima and maxima.

- $f'(x) = 3x^2 - 4x + 1$ +1

  $0 = 3x^2 - 4x + 1$

  $0 = (3x-1)(x-1)$

  $\boxed{x = \frac{1}{3}, x = 1}$ critical pts



$f'(0) = 1 > 0$

$f'(\frac{1}{2}) < 0$

$f'(2) > 0$

Incr: $(-\infty, \frac{1}{3}) \cup (1, \infty)$

Decr: $(\frac{1}{3}, 1)$

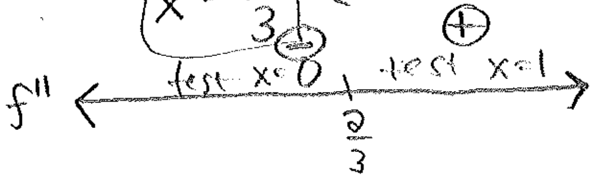
Local max at $f(\frac{1}{3}) = \frac{1}{27} - \frac{2}{9} + \frac{1}{3} - 1 = -\frac{23}{27}$

Local min at $f(1) = -1$

- $f''(x) = 6x - 4$

  $0 = 6x - 4$

  $\boxed{x = \frac{2}{3}}$ ← inflection pt



$f''(0) < 0$

$f''(1) > 0$

concave up: $(\frac{2}{3}, \infty)$

concave down: $(-\infty, \frac{2}{3})$

2) (4 points each) Evaluate the following limits (Hint: you may need to apply L'Hopital's Rule):

a) $\lim_{\theta\to\pi} \frac{\cos(\theta)+1}{\theta\sin(\theta)}$ $\underset{LH}{\overset{\frac{0}{0}}{=}} \lim_{\theta\to\pi} \frac{-\sin\theta}{\theta\cos\theta+\sin\theta} \underset{LH}{\overset{\frac{0}{0}}{=}} \lim_{\theta\to\pi} \frac{-\cos\theta}{\cos\theta-\theta\sin\theta+\cos\theta}$

$= \frac{-(-1)}{-1+(-1)} = \boxed{-\frac{1}{2}}$

b) $\lim_{x\to\infty}(e^{-2x}(x^2-x-4)) = \lim_{x\to\infty} \frac{x^2-x-4}{e^{2x}} \underset{LH}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{2x-1}{2e^{2x}}$

$0 \cdot \infty$

$\underset{LH}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{2}{4e^{2x}} = \frac{2}{\text{infinity}} = \boxed{0}$

( • Or students can write a statement about the growth of a polynomial compared with $e^{cx}$, $c>0$, and conclude that the limit is $\boxed{0}$ )

3) (8 points) Verify the Mean Value Theorem for the function ~~$f(x) = x^4 - 2x^2 + 1$~~ on the interval $[-1, 2]$.

$f(x) = x^3 + x$

Find $c$ in $(-2, 1)$ such that $f'(c) = \frac{f(1) - f(-2)}{1 - (-2)}$ :

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{2 - (-10)}{3} = 4$$

Take $f'(x) = 3x^2 + 1$ ;

(Also, $f(x)$ is continuous & differentiable everywhere, so we can apply the MVT)

set $4 = 3x^2 + 1$

$x^2 = 1$

$x = \pm 1$ ⇒ since only $-1$ is in the interval $(-1, 2)$, $c = -1$ is the desired value.

4) (12 points) Find the derivative of $y = \frac{(x+2)^3}{(x+5)(3x-4)}$ using logarithmic differentiation.

$$\ln(y) = \ln\left(\frac{(x+2)^3}{(x+5)(3x-4)}\right)$$

$$\ln(y) = 3\ln(x+2) - \ln(x+5) - \ln(3x-4)$$

$$\frac{y'}{y} = \frac{3}{x+2} - \frac{1}{x+5} - \frac{3}{3x-4}$$

$$y' = \frac{(x+2)^3}{(x+5)(3x-4)}\left(\frac{3}{x+2} - \frac{1}{x+5} - \frac{3}{3x-4}\right)$$

5) (12 points) Compute $\frac{dy}{dx}$ of $3xy^2 = y^3 - \cos(x)$ (Hint: use implicit differentiation).

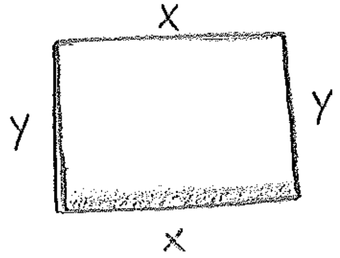
$$\frac{d}{dx}(3xy^2) = \frac{d}{dx}(y^3 - \cos(x))$$

$$3y^2 + 6xy\frac{dy}{dx} = 3y^2 \cdot \frac{dy}{dx} + \sin(x)$$

$$\frac{dy}{dx}(6xy - 3y^2) = \sin(x) - 3y^2$$

$$\frac{dy}{dx} = \frac{\sin(x) - 3y^2}{6xy - 3y^2}$$

6) (12 points) An architect plans to enclose a 750 square foot rectangular region in a botanical garden. She will use shrubs costing 10 per foot along three sides and fencing costing 5 per foot along the fourth side, Find the minimum total cost.



Area = 750 ft$^2$

$C(x,y) = 10x + 20y + 5x$

$C(x,y) = 15x + 20y$ ← objective function

$750 = xy$ ← constraint function

$\frac{750}{x} = y$

$C(x) = 15x + 20\left(\frac{750}{x}\right)$

$C(x) = 15x + \frac{15000}{x}$

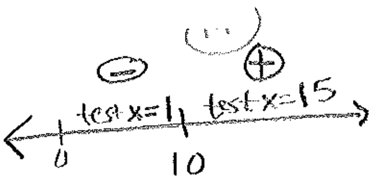
$C'(x) = 15 - \frac{15000}{x^2}$

$0 = 15 - \frac{15000}{x^2}$

$15x^2 = 15000$

$x^2 = 1000$

$x = 10 \Rightarrow y = 75$



(So C(10) is a local min.)

7) (4 points each) Compute the derivative of the following functions:

a) $f(x) = \tan(\ln(6x^4 + x^2))$

$$f'(x) = \sec^2(\ln(6x^4+x^2)) \cdot \frac{24x^3+2x}{6x^4+x^2}$$

$$f'(x) = \sec^2(\ln(6x^4+x^2)) \cdot \frac{24x^2+2}{6x^3+x}$$

b) $y = e^{sin(x)}$

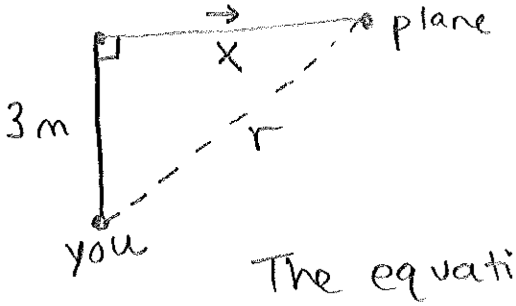
$$y' = e^{\sin(x)} \cdot \cos(x)$$

d) $g(t) = (5 + (2x^2 - 1)^3)^{1/3}$

$$g'(t) = \frac{1}{3}(5+(2x^2-1)^3)^{-2/3}(3(2x^2-1)^2 \cdot 4x)$$

$$= 4x(5+(2x^2-1)^3)^{-2/3}(2x^2-1)^2$$

8) (12 points) A plane is flying away from you at 500 mph at a height of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?



$\frac{dx}{dt} = 500$ mph

Find $\frac{dr}{dt}$ when $x=4$.

The equation $r^2 = x^2 + 3^2$ relates x & r.

Differentiate with respect to "t":

$$2r\cdot\frac{dr}{dt} = 2x\cdot\frac{dx}{dt}$$

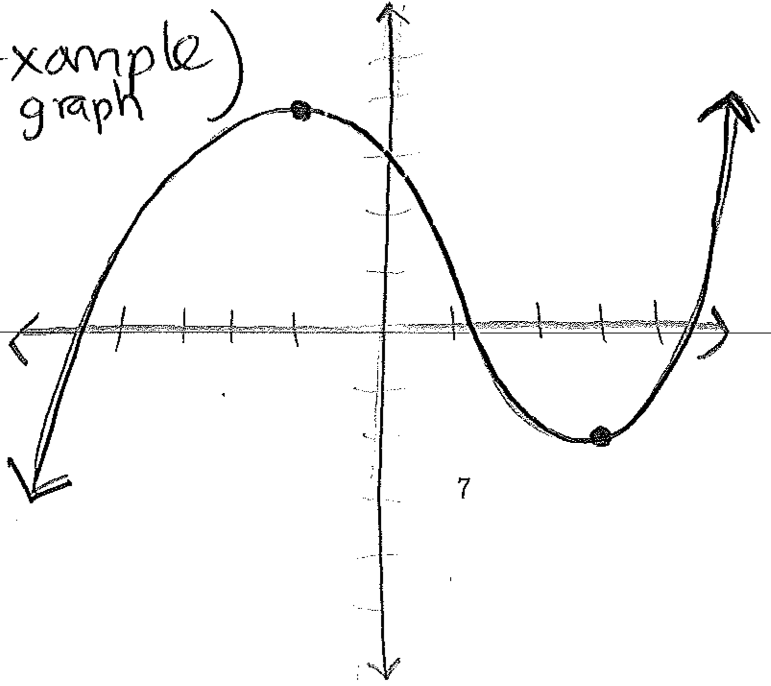
When $x=4$,

$$r = \sqrt{16+9} = \sqrt{25} = 5$$

$$\frac{dr}{dt} = \frac{x}{r}\cdot\frac{dx}{dt} = \frac{4}{5}(500) = \boxed{400 \text{ mph}}$$

9) (8 points) Sketch the graph of a function with the following features:
(i) $f'(x) > 0$ for $x < -1$ and $x > 3$ and $f'(x) < 0$ for $-1 < x < 3$, and
(ii) $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$

(Example graph)



Need
Local max at $x=-1$
Local min at $x=3$
Inflection pt at $x=1$
Increasing $(-\infty,-1) \cup (3,\infty)$
Decreasing $(-1,3)$
Concave up $(1,\infty)$
Concave down $(-\infty,1)$