SOLUTIONS

Final: MATH 220 - Calculus 1

July 28th 2017

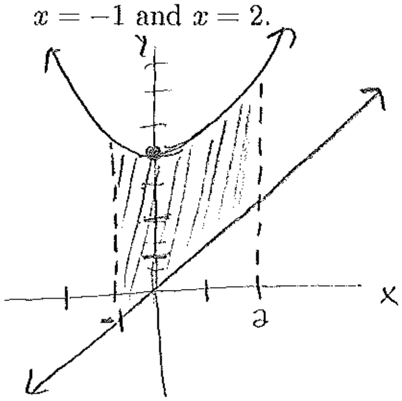
Name:

Instructor:

1	2	3	4	5	6	7	8	9	10	Total

Instructions: You have 1 hour and 15 minutes to complete this exam. Show all of your work. Calculators are not allowed.

1) (20 points) Find the area of the region enclosed by $y = x^2 + 4$, $y = x$, $x = -1$ and $x = 2$.



$$\text{Area} = \int_{-1}^{2} (x^2+4-x)\,dx$$

$$= \left.\frac{x^3}{3} + 4x - \frac{x^2}{2}\right|_{-1}^{2}$$

$$= \left(\frac{8}{3} + 8 - 2\right) - \left(-\frac{1}{3} - 4 - \frac{1}{2}\right)$$

$$= \frac{27}{2} \text{ units}^2$$

2) (5 points each) Compute the following:

a) $\frac{d}{dx}(\int_{-1}^{e^{3x}}(\ln(t)+t)dt)$

$= (\ln(e^{3x}) + e^{3x})3e^{3x}$

$= (3x + e^{3x})3e^{3x}$

b) $\int(-x^3 + 2x^{-3})dx$

$= -\frac{x^4}{4} - x^{-2} + C$

c) $\int x\cos(x^2+1)dx$ let $u = x^2+1$

$= \frac{1}{2}\int \cos(u)du$ $du = 2x\,dx$

$= \frac{1}{2}\sin(u) + C$

$= \frac{1}{2}\sin(x^2+1) + C$

d) $\int_0^1(3x^2+2x)(x^3+x^2)^3dx$ $= \int_0^2 u^3 du = \frac{u^4}{4}\Big|_0^2 = \boxed{4}$

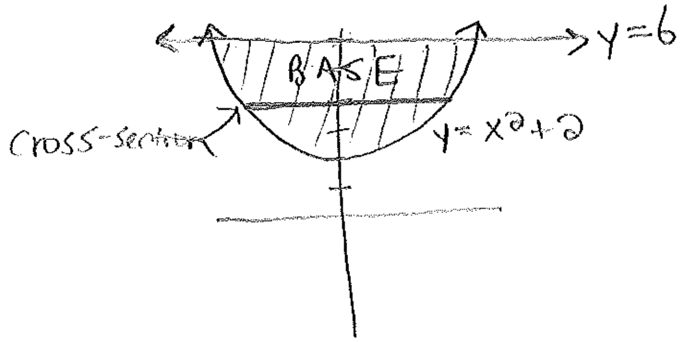
let $u = x^3+x^2$

$du = (3x^2+2x)dx$

$u(0) = 0$

$u(1) = 2$

3) (20 points) Find the volume of the solid with base bounded by $y = x^2 + 2$ and $y = 6$, where the cross-sections perpendicular to the y-axis are rectangles of height 3.



$$A(y) = (\text{base})(\text{height}) = 2(\sqrt{y-2})(3)$$

$$\text{Volume} = \int_a^b A(y)\,dy$$

$$= \int_2^6 6\sqrt{y-2}\,dy$$

$$= 6 \cdot \tfrac{2}{3}(y-2)^{3/2} \Big|_2^6$$

$$= 4\left(4^{3/2} - 0\right)$$

$$= 4(8)$$

$$= 32 \text{ units}^3$$

4) (15 points) Calculate the instantaneous rate of change for $f(x) = \frac{1}{x^2}$ at $a = -2$ using the limit definition.

$$f'(-2) = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x+2} = \lim_{x \to -2} \frac{\left(\frac{4 - x^2}{4x^2}\right)}{x+2}$$

$$= \lim_{x \to -2} \frac{\left(\frac{(2-x)\cancel{(2+x)}}{4x^2}\right)}{\cancel{x+2}} = \frac{4}{16} = \frac{1}{4}$$

5) (15 points) Find and classify the critical values of the function $f(x) = \frac{x^2}{3x-6}$.

$$f'(x) = \frac{(3x-6)(2x) - x^2(3)}{(3x-6)^2}$$

$$= \frac{3x^2 - 12x}{(3x-6)^2}$$

$0 = 3x^2 - 12x$

$0 = 3x(x-4)$

$\boxed{x = 0,\ x = 4}$ critical pts

(Although $f'(2)$ DNE, ~~x~~ $x=2$ is not in the domain of f, so cannot be a critical point)

$f'(x)$: test $x=-1$ ⊕ | 0 | test $x=1$ ⊖ | 4 | test $x=5$ ⊕

$f'(-1) > 0$

$f'(1) < 0$

$f'(5) > 0$

$\Rightarrow f(0)$ a local max

$f(4)$ a local min

6) (5 points each) Compute the following:

a) $\frac{d}{dx}((x^3+4)e^{sin(x)})$

$$=(x^3+4)e^{\sin(x)}\cos(x)+(3x^2)e^{\sin(x)}$$

b) Find $\frac{d^2y}{dx^2}$ for $y = e^{-x} + \ln(x)$.

$$\frac{dy}{dx}=y'=-e^{-x}+\frac{1}{x}$$

$$\frac{d^2y}{dx^2}=y''=e^{-x}-\frac{1}{x^2}$$

c) Find $\frac{dy}{dx}$ for $y^3 = cos(y) + x^2$.

$$\frac{d}{dx}(y^3)=\frac{d}{dx}(\cos(y)+x^2)$$

$$3y^2\cdot\frac{dy}{dx}=-\sin(y)\cdot\frac{dy}{dx}+2x$$

$$\frac{dy}{dx}=\frac{2x}{3y^2-\sin(y))}$$

d) $\frac{d}{dz}((z-3z^{-1})^2)$

$$=2(z+3z^{-1})(1+3z^{-2})$$

7) (4 points each) Evaluate the following limits:

a) $\lim_{x\to 0} \frac{\cancel{x}\sin(x)}{\cos(x)-1}$

$\frac{0}{0}$ ||

$$\lim_{x\to 0} \frac{x\cos(x)+\sin(x)}{-\sin(x)} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-x\sin x+\cos(x)+\cos(x)}{-\cos(x)}$$

$$= \frac{2}{-1} = -2$$

b) $\lim_{x\to -3} \frac{\ln(x+5)}{x+1} = \frac{\ln(2)}{-2}$

$$= \ln\left(\frac{1}{\sqrt{2}}\right)$$

8) (10 points) State The Fundamental Theorem of Calculus Pt I, Pt II, or the Intermediate Value Theorem.

FTC Pt I: let $f(x)$ be continuous on $[a,b]$ and $F'(x) = f(x)$. Then $\int_a^b f(x)dx = F(b) - F(a)$.

FTC. Pt II: let $f(x)$ be continuous on an open interval, I, and let ~~x~~ a be in I. Define $A(x) = \int_a^x f(t)dt$. Then $A(x)$ is an antiderivative of $f(x)$ / $A'(x) = f(x)$ / $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$ $\{$Also, $A(a) = 0\}$

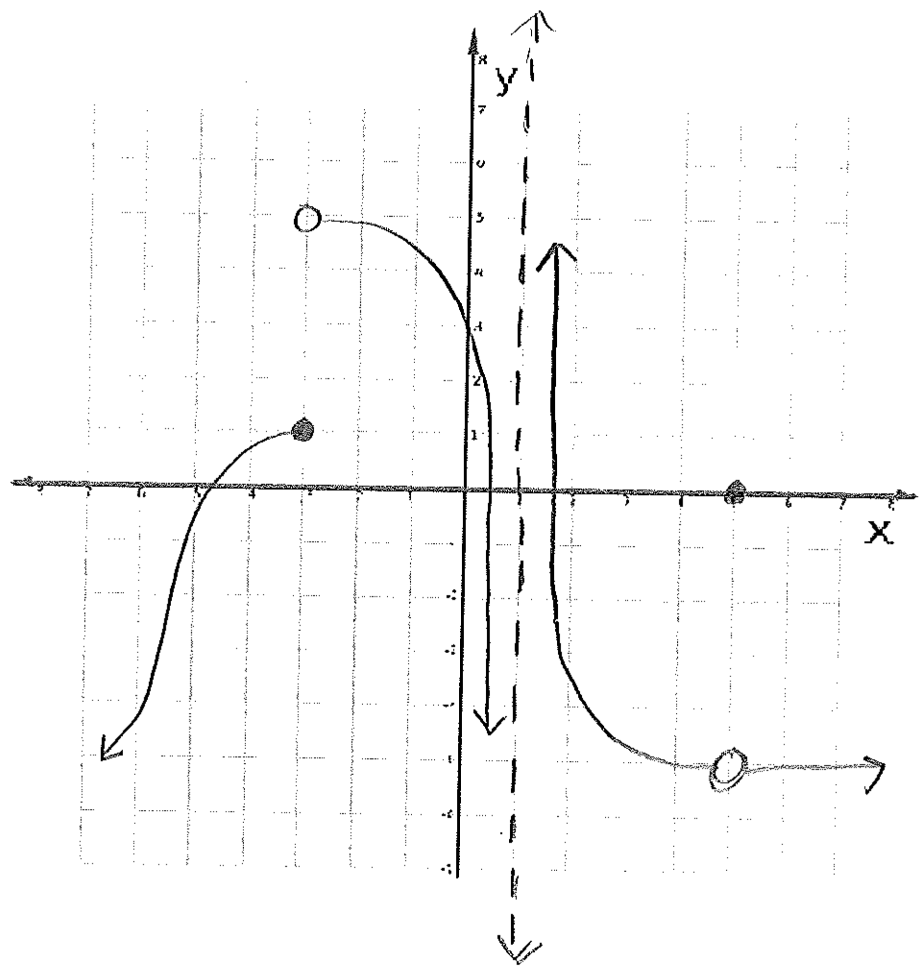
IVT: let $f(x)$ be continuous on $[a,b]$. Then for any value M between $f(a)$ and $f(b)$ there is a value $x=c$ in (a,b) such that $f(c)=M$.

9) (12 points) Sketch the graph of a function $y = f(x)$ satisfying all of the following criteria:

(i) Jump discontinuity at $x = -3$ such that f(x) is left-continuous at $x = -3$.

(ii) $\lim_{x\to 5} f(x) = -4$ and $f(5) = 0$.

(iii) $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = -\infty$



10) (10 points) Identify the graphs of $f(x)$, $f'(x)$, and $f''(x)$: