Name Solutions Rec. Instr.

Signature Rec. Time

Math 220

Final Exam

December 12, 2018

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		15	8		6
2		15	9		6
3		6	10		6
4		6	11		6
5		6	12		5
6		6	13		5
7		6	14		6

Total Score:

1. (3 points each) Evaluate the following:

A. $\lim_{x\to 1} \frac{\ln(x)}{x-1} =$ $\underset{\left(\frac{0}{0}\right)}{\overset{}{\text{LH}}}$ $\lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1$

B. $\int \left(\frac{2}{x} - \sqrt{x}\right) dx =$ $2\ln|x| - \frac{2}{3}x^{3/2} + C$

C. $\frac{d}{dx}\int_x^5 e^{\sin(t)}\, dt =$ $\frac{d}{dx}\left(-\int_5^x e^{\sin(t)} dt\right) = -e^{\sin(x)}$

D. $\frac{d}{dx}\left(\frac{\tan(x)}{\ln(x)+3}\right) =$ $\frac{\sec^2(x)\cdot(\ln(x)+3) - \tan(x)\cdot\frac{1}{x}}{(\ln(x)+3)^2}$

E. $\frac{d}{dx}\left(\sin(x^2)\cdot\arctan(x)\right) =$ $\cos(x^2)\cdot 2x\cdot\arctan(x) + \sin(x^2)\cdot\frac{1}{1+x^2}$

2. (5 points each) Find the following:

A. $\int \frac{\cos(\ln(x))}{x}\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(\ln(x)) + C$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

B. $\frac{dy}{dx}$ if $x^3 + xy + y^4 = 5$

$\frac{d}{dx}[x^3 + xy + y^4] = \frac{d}{dx}[5]$

$3x^2 + y + x \cdot \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 0$

$x \cdot \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = -3x^2 - y$

$(x + 4y^3)\frac{dy}{dx} = -3x^2 - y$

$\frac{dy}{dx} = \frac{-3x^2 - y}{x + 4y^3}$

C. $f'(x)$ if $f(x) = x^{3x}$

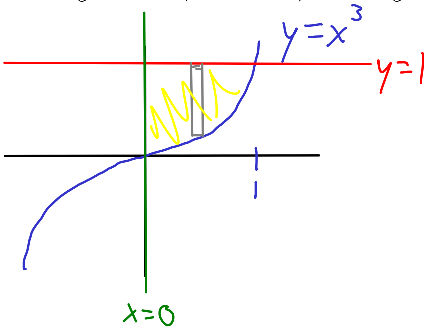
$\ln(f(x)) = \ln(x^{3x}) = 3x\ln(x)$

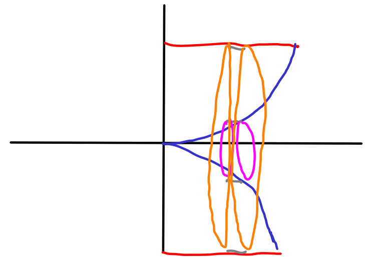
$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}[3x\ln(x)]$

$\frac{f'(x)}{f(x)} = 3\ln(x) + 3x \cdot \frac{1}{x} = 3\ln(x) + 3$

$f'(x) = f(x) \cdot (3\ln(x) + 3) = x^{3x} \cdot (3\ln(x) + 3)$

3. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $x = 0$, and $y = 1$ around the x-axis.



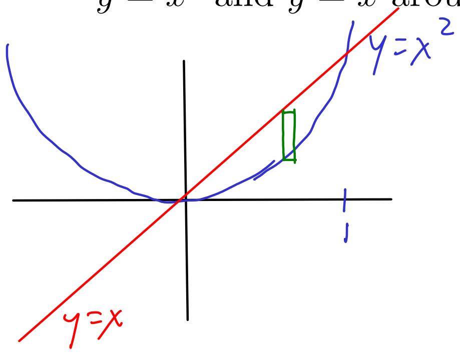


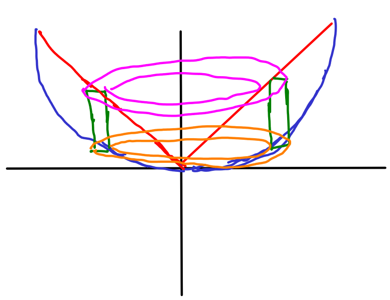
$$\text{Volume} = \int_0^1 \pi \cdot 1^2 \, dx - \int_0^1 \pi \cdot (x^3)^2 \, dx = \int_0^1 (\pi - \pi x^6) \, dx$$

$$= \left(\pi x - \frac{\pi x^7}{7}\right)\Big|_0^1 = \left(\pi \cdot 1 - \frac{\pi \cdot 1^7}{7}\right) - \left(\pi \cdot 0 - \frac{\pi \cdot 0^7}{7}\right)$$

$$= \pi - \frac{\pi}{7} = \frac{6\pi}{7}$$

4. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = x$ around the y-axis.





$$\text{Volume} = \int_0^1 2\pi x \cdot (x - x^2) \, dx$$

$$= \int_0^1 (2\pi x^2 - 2\pi x^3) \, dx$$

$$= \left(\frac{2\pi x^3}{3} - \frac{\pi x^4}{2}\right)\Big|_0^1$$

$$= \left(\frac{2\pi \cdot 1^3}{3} - \frac{\pi \cdot 1^4}{2}\right) - \left(\frac{2\pi \cdot 0^3}{3} - \frac{\pi \cdot 0^4}{2}\right)$$

$$= \frac{2\pi}{3} - \frac{\pi}{2}$$

$$= \frac{\pi}{6}$$

5. (6 points) Suppose that a particle has position $s(t)$ feet at time t seconds and a velocity function $s'(t) = t \cdot \sin(\pi t^2)$ ft/s. Find the displacement (change in position) from time $t = 0$ seconds to time $t = 1$ seconds. (Include units with your answer.)

$$s(1) - s(0) = \int_0^1 s'(t)\,dt = \int_0^1 t \cdot \sin(\pi t^2)\,dt$$

$$= \int_0^{\pi} \sin(u) \cdot \frac{du}{2\pi} = -\frac{\cos(u)}{2\pi}\Big|_0^{\pi}$$

$$= -\frac{\cos(\pi)}{2\pi} + \frac{\cos(0)}{2\pi}$$

$$= \frac{1}{2\pi} + \frac{1}{2\pi}$$

$$= \frac{1}{\pi} \text{ ft}$$

$u = \pi t^2$

$\frac{du}{dt} = 2\pi t$

$\frac{du}{2\pi} = t\,dt$

t	u
1	π
0	0

6. (6 points) Use a linearization of $u(x) = \ln(x)$ at $x = 1$ to approximate $\ln(.9)$.

$u'(x) = \frac{1}{x}$.

$L(x) = u(1) + u'(1) \cdot (x-1) = \ln(1) + \frac{1}{1}(x-1) = x - 1$

$\ln(.9) = u(.9) \approx L(.9) = .9 - 1 = -.1$

.9 is close to 1

7. (6 points) Find the absolute minimum and maximum of $w(x) = x - \sqrt{x}$ on the interval $[0, 4]$.

$w'(x) = 1 - \frac{1}{2\sqrt{x}}$ is defined for $x > 0$.

$w'(x) = 0$ when $1 = \frac{1}{2\sqrt{x}} \iff \sqrt{x} = \frac{1}{2} \iff x = \frac{1}{4}$

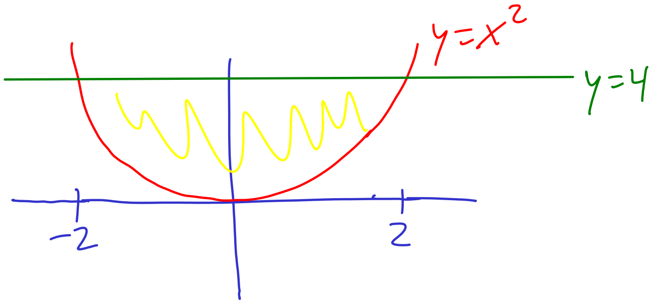
$w(0) = 0 - \sqrt{0} = 0$

$w(\frac{1}{4}) = \frac{1}{4} - \sqrt{\frac{1}{4}} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$w(4) = 4 - \sqrt{4} = 4 - 2 = 2$

On $[0,4]$, $w(x)$ has an absolute max $w(4) = 2$ and an absolute min $w(\frac{1}{4}) = -\frac{1}{4}$.

8. (6 points) Find the area between the curves $y = 4$ and $y = x^2$. (You do not need to simplify your final **numeric** answer.)

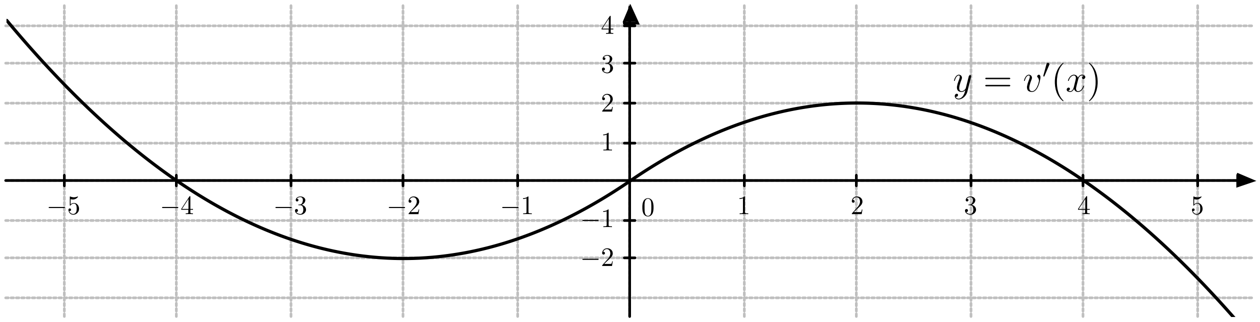


$$\text{AREA} = \int_{-2}^{2} |4 - x^2| \, dx = \int_{-2}^{2} (4 - x^2) \, dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2}$$

$$= \left(4 \cdot 2 - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= \frac{16}{3} + \frac{16}{3} = \frac{32}{3}$$



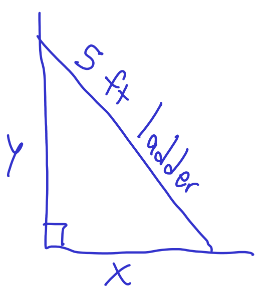
9. (2 points each) $y = v'(x)$ is plotted above. Find:

A. Interval(s) where $v(x)$ is increasing: $(-\infty,-4),\ (0,4)$ decreasing: $(-4,0),\ (4,\infty)$

B. x-coordinate(s) where $v(x)$ has a local max: $-4,\ 4$ local min: 0

C. Interval(s) where $v(x)$ is concave up: $(-2,2)$ concave down: $(-\infty,-2),\ (2,\infty)$

10. (6 points) A 5-foot ladder rests against the wall. The bottom of the ladder slides away from the wall at a rate of 2 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 ft from the wall? (Include units with your answer.)



Want: $\frac{dy}{dt}$ when $x = 3$ ft

Know: $\frac{dx}{dt} = 2$ ft/s

$$x^2 + y^2 = 5^2$$

$$\frac{d}{dt}\left[x^2+y^2\right] = \frac{d}{dt}\left[5^2\right]$$

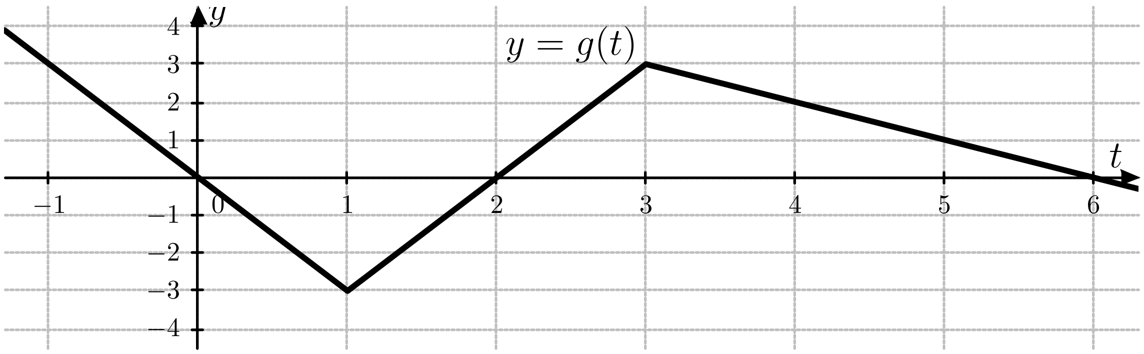
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2y\frac{dy}{dt} = -2x \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-2x}{2y} \cdot \frac{dx}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

When $x = 3$ ft, $y = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$ ft, and

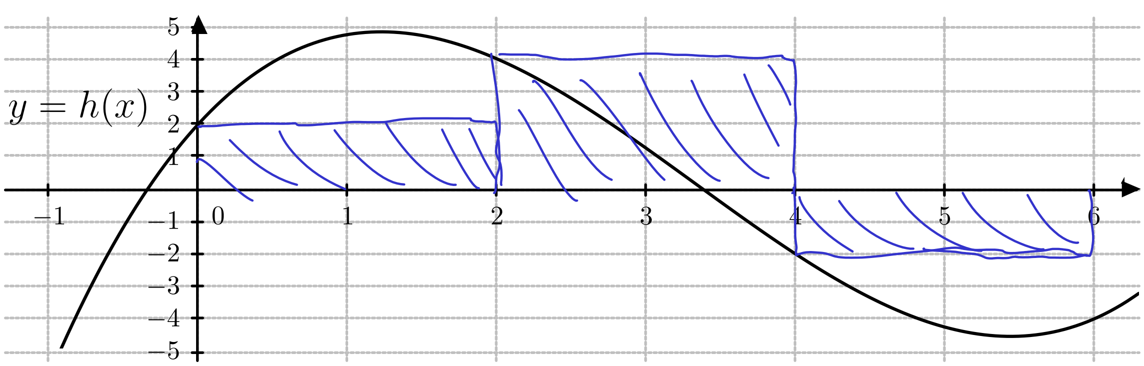
$\frac{dy}{dt} = -\frac{3}{4} \cdot 2 = -\frac{3}{2}$ ft/s. The ladder is sliding down the wall at a rate of $\frac{3}{2}$ ft/s.



11. (3 points each) $y = g(t)$ is plotted above. Let $A(x) = \displaystyle\int_0^x g(t)\,dt$. Find the following quantities.

A. $A(2) = \int_0^2 g(t)\,dt = -\frac{1}{2}\cdot 2\cdot 3 = -3$

B. $A'(4) = g(4) = 2$ because $A'(x) = \frac{d}{dx}\int_0^x g(t)\,dt = g(x)$



12. (5 points) Estimate $\displaystyle\int_0^6 h(x)\,dx$ by computing L_3, the Left-Endpoint Approximation with 3 subintervals. Also, illustrate the corresponding rectangles on the graph above.

$\Delta x = \frac{6-0}{3} = 2$

$$\int_0^6 h(x)\,dx \approx L_3 = h(0)\Delta x + h(2)\Delta x + h(4)\Delta x$$
$$= 2\cdot 2 + 4\cdot 2 + (-2)\cdot 2$$
$$= 8$$

13. (5 points) Using the **limit definition of the derivative**, find $f'(2)$ if $f(x) = x^2$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+4h+h^2-\cancel{4}}{h} = \lim_{h\to 0} \frac{4h+h^2}{h}$$

$$= \lim_{h\to 0} (4+h) = 4+0 = 4$$

14. (6 points) When a company charges x dollars per chair, it makes a total profit of $P(x) = -2x^2 + 200x - 50$ dollars. If the company wants to maximize total profit, what should it charge per chair? (Make sure to justify why your answer corresponds to the absolute maximum.)

$P'(x) = -4x + 200$ is always defined

$P'(x) = 0$ when $4x = 200 \iff x = \frac{200}{4} = 50$

1st Derivative Justification

sign of $P'(x)$: $+$ before 50, $-$ after 50

$P(x)$ is increasing for $x < 50$ and decreasing for $x > 50$

2nd Derivative Justification

$P''(x) = -4 < 0$ for all x

$P(x)$ is always concave down.

$P(x)$ is maximized by charging $x = \$50$ per chair.