Name Solutions Rec. Instr.

Signature Rec. Time

Math 220
Final Exam
December 18, 2019

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		15	8		6
2		15	9		6
3		6	10		6
4		6	11		6
5		6	12		5
6		6	13		5
7		6	14		6

Total Score:

1. (3 points each) Evaluate the following:

A. $$\lim_{x\to\infty}\frac{3e^x+x}{7x-5e^x} \underset{\left(\substack{\text{LH}\\ \frac{\infty}{-\infty}}\right)}{=} \lim_{x\to\infty}\frac{3e^x+1}{7-5e^x} \underset{\left(\substack{\text{LH}\\ \frac{\infty}{-\infty}}\right)}{=} \lim_{x\to\infty}\frac{3e^x}{-5e^x} = \lim_{x\to\infty} -\frac{3}{5} = -\frac{3}{5}$$

B. $$\int\left(2\sqrt{x}-\frac{1}{x^2}\right)dx = \frac{4}{3}x^{3/2}+\frac{1}{x}+C$$

C. $$\frac{d}{dx}\int_x^7 \sin(t^5)\cos(t^2+1)\,dt = -\frac{d}{dx}\int_7^x \sin(t^5)\cos(t^2+1)\,dt$$

$$= \sin(x^5)\cos(x^2+1)$$

D. $$\frac{d}{dx}\left(\frac{\cos(x^7)}{\ln(x)+3}\right) = \frac{-\sin(x^7)\cdot 7x^6(\ln(x)+3)-\cos(x^7)\cdot\frac{1}{x}}{(\ln(x)+3)^2}$$

E. $$\frac{d}{dx}\left(\tan(x^2)\cdot\arctan(x)\right) = \sec^2(x^2)\cdot 2x\cdot\arctan(x)+\tan(x^2)\cdot\frac{1}{1+x^2}$$

2. (5 points each) Find the following:

A. $\int \frac{\sec^2(2\theta)}{\tan^5(2\theta)}\, d\theta$ $= \int \frac{1}{u^5} \cdot \frac{du}{2} = \int \frac{1}{2} u^{-5} du = \frac{1}{2} \cdot \frac{1}{-4} u^{-4} + C$

$$= -\frac{1}{8u^4} + C = \frac{-1}{8\tan^4(2\theta)} + C$$

$$\boxed{\begin{aligned} &u = \tan(2\theta) \\ &\frac{du}{d\theta} = \sec^2(2\theta)\cdot 2 \\ &\frac{du}{2} = \sec^2(2\theta)\, d\theta \end{aligned}}$$

B. $\frac{dy}{dx}$ if $y^4 + xy = x^3 - x + 2$

$$\frac{d}{dx}\left[y^4 + xy\right] = \frac{d}{dx}\left[x^3 - x + 2\right]$$

$$4y^3 \frac{dy}{dx} + y + x\frac{dy}{dx} = 3x^2 - 1$$

$$4y^3 \frac{dy}{dx} + x\frac{dy}{dx} = 3x^2 - 1 - y$$

$$(4y^3 + x)\frac{dy}{dx} = 3x^2 - 1 - y$$

$$\frac{dy}{dx} = \frac{3x^2 - 1 - y}{4y^3 + x}$$

C. $k'(x)$ if $k(x) = \frac{x^x}{\sin^7(x)}$

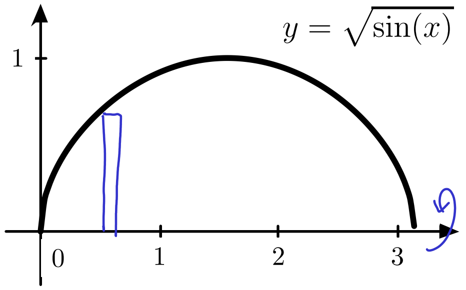
$$\ln(k(x)) = \ln\left(\frac{x^x}{\sin^7(x)}\right) = \ln(x^x) - \ln(\sin^7(x)) = x \cdot \ln(x) - 7\ln(\sin(x))]$$

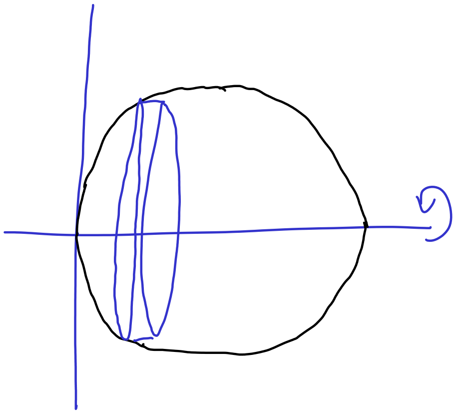
$$\frac{d}{dx}\ln(k(x)) = \frac{d}{dx}\left(x \cdot \ln(x) - 7\ln(\sin(x))\right)$$

$$\frac{k'(x)}{k(x)} = 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 7 \cdot \frac{\cos(x)}{\sin(x)} = \ln(x) + 1 - \frac{7\cos(x)}{\sin(x)}$$

$$k'(x) = k(x)\left(\ln(x) + 1 - \frac{7\cos(x)}{\sin(x)}\right) = \frac{x^x}{\sin^7(x)}\left(\ln(x) + 1 - \frac{7\cos(x)}{\sin(x)}\right)$$

3. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{\sin(x)}$ and $y = 0$ between $x = 0$ and $x = \pi$ around the x-axis.

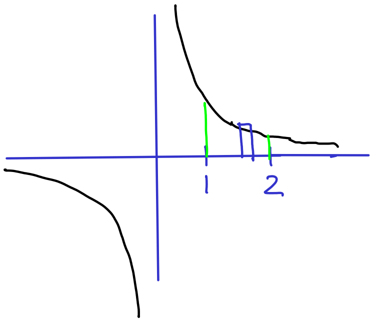


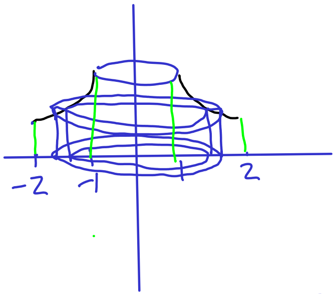


$$\text{Volume} = \int_0^\pi \pi\left(\sqrt{\sin(x)}\right)^2 dx = \int_0^\pi \pi \sin(x)\,dx = -\pi\cos(x)\Big|_0^\pi$$

$$= -\pi\cdot\cos(\pi) + \pi\cos(0) = 2\pi$$

4. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, and $x = 2$ around the y-axis.





$$\text{Volume} = \int_1^2 2\pi x\cdot\frac{1}{x}\,dx = \int_1^2 2\pi\,dx = 2\pi x\Big|_1^2$$

$$= 2\pi\cdot 2 - 2\pi\cdot 1 = 2\pi$$

5. (6 points) Suppose that a particle has position $s(t)$ feet at time t seconds and a velocity function $s'(t) = t \cdot e^{-t^2}$ ft/s. Find the displacement (change in position) from time $t = 0$ seconds to time $t = 1$ seconds. (Include units with your answer.)

displacement
from $t=0$ s
to $t=1$ s

$$= s(1) - s(0) = \int_0^1 t e^{-t^2}\,dt$$

$$= \int_0^{-1} -\tfrac{1}{2} e^u\,du = -\tfrac{1}{2} e^u \Big|_0^{-1} = -\tfrac{1}{2} e^{-1} + \tfrac{1}{2} e^0$$

$$= \frac{1}{2} - \frac{1}{2e} \text{ ft}$$

$u = -t^2$

$\frac{du}{dt} = -2t\,dt$

$-\frac{du}{2} = t\,dt$

t	u
1	-1
0	0

6. (6 points) Use a linearization of $u(x) = \sqrt{x}$ at $x = 25$ to approximate $\sqrt{26}$.

$u'(x) = \frac{1}{2\sqrt{x}}$. The linearization of $u(x)$ at $x = 25$ is

$$L(x) = u(25) + u'(25)(x-25) = \sqrt{25} + \frac{1}{2\sqrt{25}}(x-25)$$

$$L(x) = 5 + \frac{x-25}{10}$$

$$\sqrt{26} = u(26) \underset{\uparrow}{\approx} L(26) = 5 + \frac{26-25}{10} = 5.1$$

26 is close
to 25

7. (6 points) Find the absolute minimum and maximum of $w(x) = x + \cos(x)$ on the interval $[0, \pi]$.

$w'(x) = 1 - \sin(x)$ is defined everywhere,

$0 = w'(x) = 1 - \sin(x)$ for $0 \le x \le \pi$ only when $x = \frac{\pi}{2}$.

$w(0) = 0 + \cos(0) = 1$

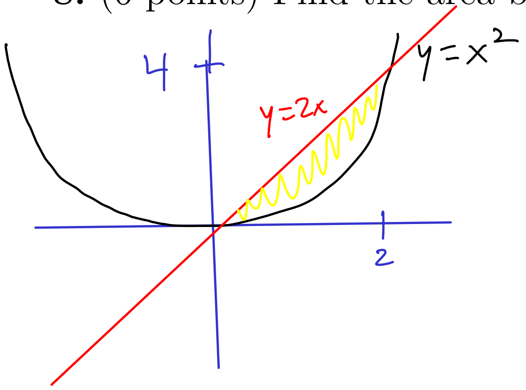
$w(\frac{\pi}{2}) = \frac{\pi}{2} + \cos(\frac{\pi}{2}) = \frac{\pi}{2}$

$w(\pi) = \pi + \cos(\pi) = \pi - 1$

On $[0, \pi]$, $w(x)$ has an absolute min $w(0) = 1$

and an absolute max $w(\pi) = \pi - 1$.

8. (6 points) Find the area between the curves $y = 2x$ and $y = x^2$.

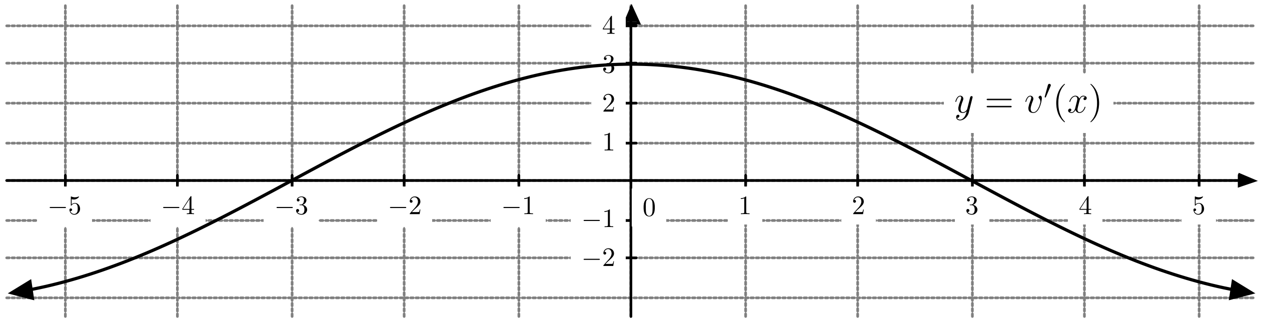


$$\text{Area} = \int_0^2 (2x - x^2)\, dx$$

$$= \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= \left(2^2 - \frac{1}{3} \cdot 2^3\right) - \left(0^2 - \frac{1}{3} \cdot 0^3\right)$$

$$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$



9. (2 points each) $y = v'(x)$ is plotted above. Find:

A. Interval(s) where $v(x)$ is increasing: $(-3, 3)$ decreasing: $(-\infty,-3), (3,\infty)$

B. x-coordinate(s) where $v(x)$ has a local max: $x=3$ local min: $x=-3$

C. Interval(s) where $v(x)$ is concave up: $(-\infty, 0)$ concave down: $(0,\infty)$

10. (6 points) Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and the volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant, the volume is 30 cm^3, the pressure is 100 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume changing at this instant? (Include units with your answer.)

Want: $\frac{dV}{dt}$ when $V=30\text{cm}^3$, $P=100\text{kPa}$, $\frac{dP}{dt}=20 \text{ kPa/min}$

$PV=C$, where C is a constant.

$$\frac{d}{dt}[PV]=\frac{d}{dt}[C]$$

$$\frac{dP}{dt}\cdot V + P\cdot\frac{dV}{dt} = 0$$

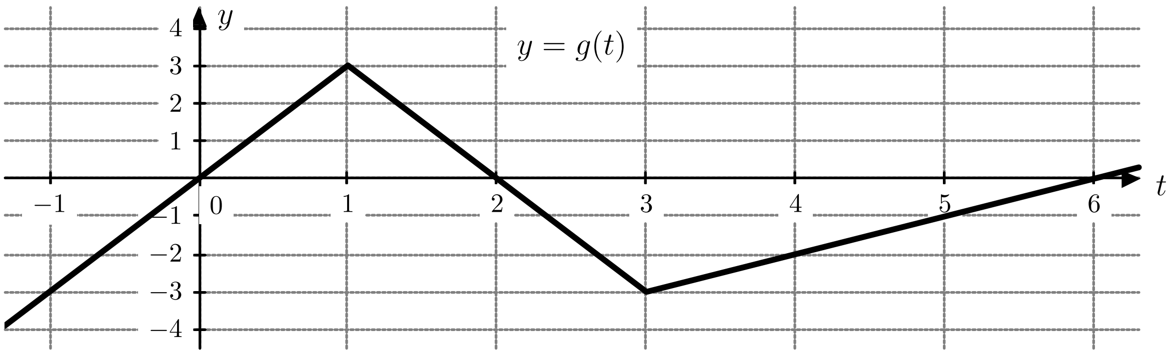
$$P\cdot\frac{dV}{dt} = -\frac{dP}{dt}\cdot V$$

$$\frac{dV}{dt} = -\frac{dP}{dt}\cdot\frac{V}{P}$$

When $V=30\text{cm}^3$, $P=100\text{kPa}$, and $\frac{dP}{dt}=20 \text{ kPa/min}$

$$\frac{dV}{dt} = -20\cdot\frac{30}{100} = \frac{-600}{100} = -6 \text{ cm}^3/\text{min}$$

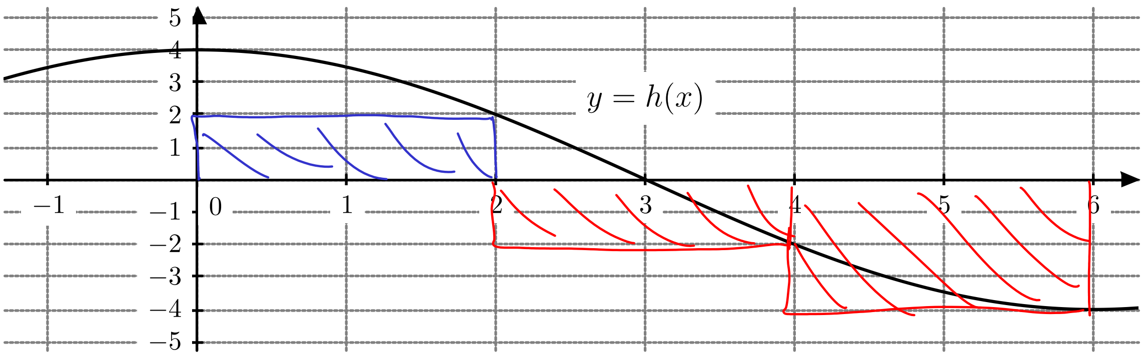
(decreasing at a rate of 6 cm^3/min)



11. (3 points each) $y = g(t)$ is plotted above. Let $A(x) = \int_0^x g(t)\,dt$. Find the following quantities.

A. $A(3) = \frac{1}{2}\cdot 2\cdot 3 - \frac{1}{2}\cdot 1\cdot 3 = 3 - \frac{3}{2} = \frac{3}{2}$

B. $A'(3) = g(3) = -3$ because $A'(x) = \frac{d}{dx}\int_0^x g(t)\,dt = g(x)$.



12. (5 points) Estimate $\int_0^6 h(x)\,dx$ by computing R_3, the Right-Endpoint Approximation with 3 subintervals. Also, illustrate the corresponding rectangles on the graph above.

$\Delta x = \frac{6-0}{3} = 2.$

$\int_0^6 h(x)\,dx \approx R_3 = f(2)\Delta x + f(4)\Delta x + f(6)\Delta x$

$= 2\cdot 2 - 2\cdot 2 - 4\cdot 2 = -8$

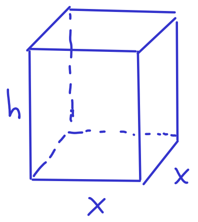
13. (5 points) Using the **limit definition of the derivative**, find $f'(2)$ if $f(x) = x^2 + 3x$.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 + 3(2+h) - (2^2 + 3 \cdot 2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 + \cancel{6} + 3h - \cancel{4} - \cancel{6}}{h} = \lim_{h \to 0} \frac{7h + h^3}{h}$$

$$= \lim_{h \to 0} (7 + h) = 7 + 0 = 0$$

14. (6 points) A rectangular open-topped box is to have a square base and volume 8 ft^3. If material for the base costs \$2 per ft^2 and material for the sides costs \$1 per ft^2, what dimensions minimize the cost of the box? (Justify why your answer is an absolute minimum, and include units in your answer.)



Minimize Cost $C = 2x^2 + 4xh$

$8 = x^2 h$ so $h = \frac{8}{x^2}$

Minimize $C(x) = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}$ on $(0, \infty)$

$C'(x) = 4x - \frac{32}{x^2}$ is defined on $(0, \infty)$.

$0 = C'(x) \Rightarrow 4x = \frac{32}{x^2} \Rightarrow 4x^3 = 32$

$\Rightarrow x^3 = 8 \Rightarrow x = 2$

1st Derivative Justification

C (x) ↘ ↗

Sign of C'(x) – +

0 2

2nd DerivatNe Justification

$C''(x) = 4 + \frac{64}{x^3} > 0$ on $(0, \infty)$

so $C(x)$ is CU on $(0, \infty)$

The cost is minimized when $x = 2$ ft and $h = \frac{8}{2^2} = 2$ ft

so a 2 ft × 2 ft × 2 ft box.