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Math 220

Final Exam

May 11, 2022

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		15	8		6
2		18	9		6
3		5	10		5
4		5	11		6
5		6	12		6
6		4	13		6
7		6	14		6

Total Score:

1. (3 points each) Evaluate the following. You do not need to simplify your final answers.

A. $\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1$

(LH)

B. $\int \left(\frac{1}{t^2} - \sqrt{t}\right) dt = -\frac{1}{t} - \frac{2}{3} t^{3/2} + C$

C. $\frac{d}{dx} \int_x^7 \cos(\sin(t))\, dt = -\frac{d}{dx} \int_7^x \cos(\sin(t))\, dt = -\cos(\sin(x))$

D. $\frac{d}{dx}\left(\frac{\cos(x^2)}{\sin(x)+e^x}\right) = \frac{-\sin(x^2)\cdot 2x\cdot(\sin(x)+e^x) - \cos(x^2)(\cos(x)+e^x)}{(\sin(x)+e^x)^2}$

E. $\frac{d}{dx}\left(\sqrt{x^3+1}\cdot \arctan(x^2)\right) = \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2 \cdot \arctan(x^2) + \sqrt{x^3+1} \cdot \frac{1}{1+(x^2)^2} \cdot 2x$

$= \frac{3x^2\arctan(x^2)}{2\sqrt{x^3+1}} + \frac{2x\sqrt{x^3+1}}{1+x^4}$

2. (6 points each) Find the following:

A. $\int x\sqrt{x-1}\,dx$ $= \int (u+1)\sqrt{u}\,du = \int (u^{3/2}+\sqrt{u})\,du$

$u = x-1 \quad u+1 = x$

$du = dx$

$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$

B. $\frac{dy}{dx}$ if $y^4 + xy = x^3 - x + 2$

$\frac{d}{dx}(y^4 + xy) = \frac{d}{dx}(x^3 - x + 2)$

$4y^3 \cdot \frac{dy}{dx} + y + x\frac{dy}{dx} = 3x^2 - 1$

$4y^3\frac{dy}{dx} + x\frac{dy}{dx} = 3x^2 - 1 - y$

$(4y^3 + x)\frac{dy}{dx} = 3x^2 - 1 - y$

$\frac{dy}{dx} = \frac{3x^2 - 1 - y}{4y^3 + x}$

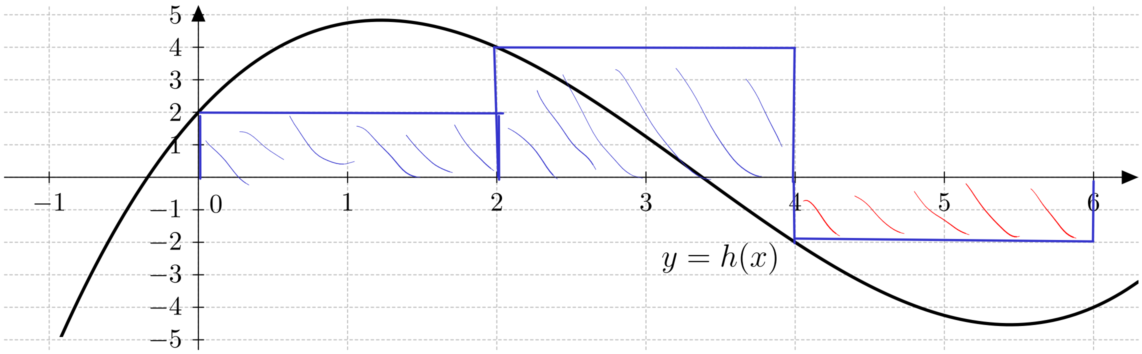
C. $k'(x)$ if $k(x) = x^{3x}$

$\ln(k(x)) = \ln(x^{3x}) = 3x\ln(x)$

$\frac{d}{dx}\ln(k(x)) = \frac{d}{dx}(3x\ln(x))$

$\frac{k'(x)}{k(x)} = 3\ln(x) + 3x \cdot \frac{1}{x} = 3\ln(x) + 3$

$k'(x) = k(x)(3\ln(x) + 3) = x^{3x}(3\ln(x) + 3)$



3. (5 points) $y = h(x)$ is plotted above. Estimate $\int_0^6 h(x)\,dx$ by using a Riemann sum with $n = 3$ subintervals, taking the sampling points to be left endpoints (the Left-Endpoint Approximation L_3). Also, illustrate the rectangles on the graph above.

$\Delta x = \frac{6-0}{3} = 2$

$\int_0^6 h(x)\,dx \approx L_3 = 2\cdot 2 + 4\cdot 2 - 2\cdot 2 = 8$

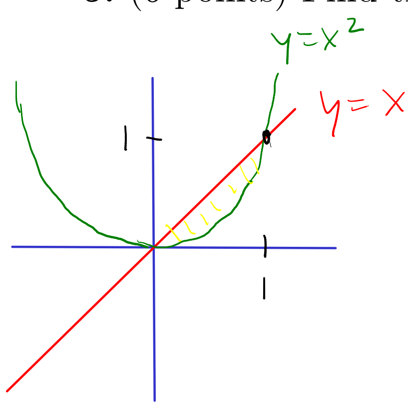
4. (5 points) Find $f(x)$ if $f'(x) = e^x + 2$ and $f(0) = 7$.

$f(x) = e^x + 2x + C$ for some constant C

$7 = f(0) = e^0 + C = 1 + C$ so $C = 6$

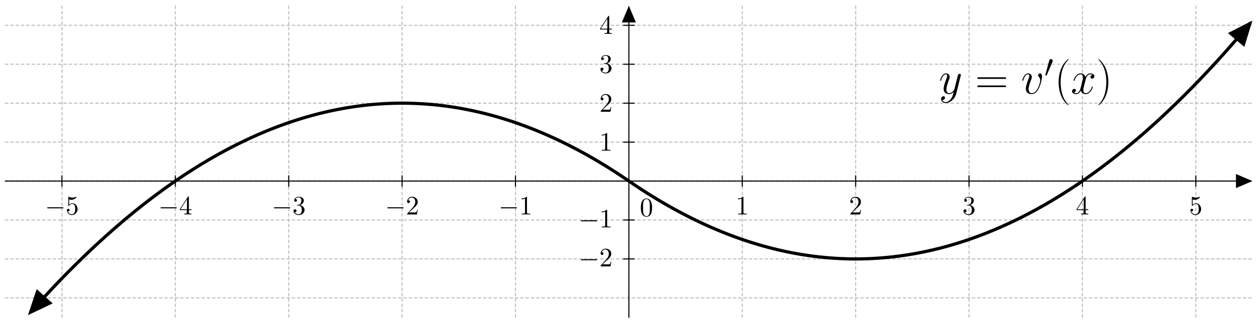
$f(x) = e^x + 2x + 6$

5. (6 points) Find the area between the curves $y = x$ and $y = x^2$.



$\text{AREA} = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$

$= \left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$



6. (1 point each) $y = v'(x)$ is plotted above. Find:

A. Interval(s) where $v(x)$ is increasing: $(-4,0),\ (4,\infty)$ decreasing: $(-\infty,-4),\ (0,4)$

B. x-coordinate(s) where $v(x)$ has a local max: 0 local min: $-4,\ 4$

C. Interval(s) where $v(x)$ is concave up: $(-\infty,-2),\ (2,\infty)$ concave down: $(-2,2)$

D. x-coordinate(s) where $v(x)$ has an inflection point: $-2,\ 2$

7. (6 points) When a company charges x dollars per backpack, it makes a total profit of $P(x) = -2x^2 + 200x - 50$ dollars. If the company wants to maximize total profit, what should it charge per backpack? (Make sure to justify why your answer corresponds to the absolute maximum.)

$P'(x) = -4x + 200$ is always defined.

$P'(x) = 0 \iff 4x = 200 \iff x = 50$

$P(x)$: $\nearrow$ $\searrow$

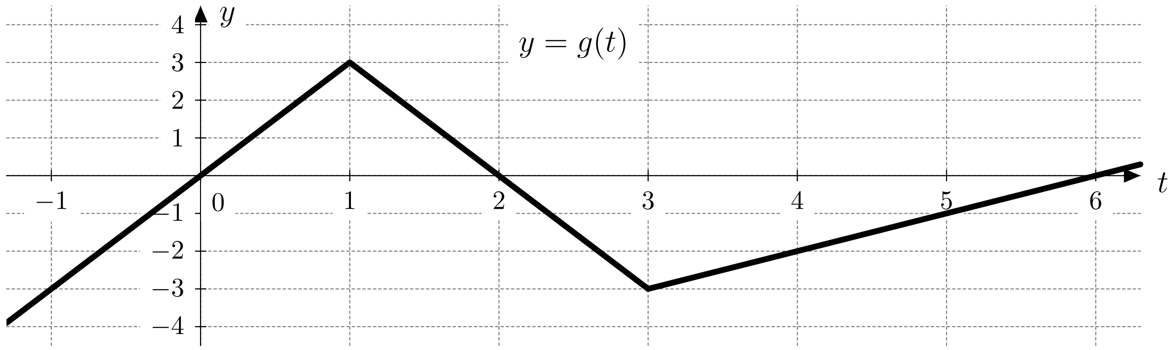
sign of $P'(x)$: $+$ on $(0, 50)$, $-$ after 50

or

$P''(x) = -4 < 0$

$P(x)$ is alway concave down

$P(x)$ is maximized when the company charges $x = \$50$ per backpack.

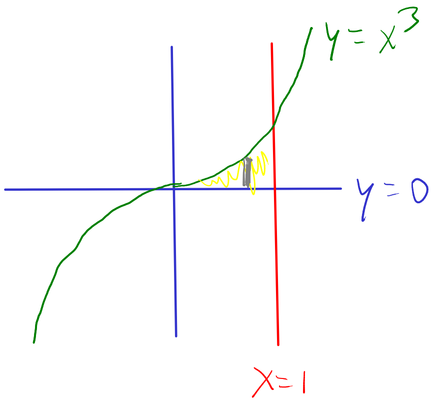


8. (3 points each) $y = g(t)$ is plotted above. Let $A(x) = \displaystyle\int_0^x g(t)\,dt$. Find the following quantities.

A. $A(3) =$ $\int_0^3 g(t)\,dt = \frac{1}{2}\cdot 2\cdot 3 - \frac{1}{2}\cdot 1\cdot 3 = 3 - \frac{3}{2} = \frac{3}{2}$

B. $A'(3) =$ $g(3) = -3$ because $A'(x) = \frac{d}{dx}\int_0^x g(t)\,dt = g(x)$.

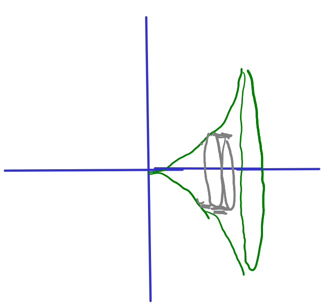
9. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 0$, and $x = 1$ around the x-axis.



$$\text{Volume} = \int_0^1 \pi (x^3)^2\,dx = \int_0^1 \pi x^6\,dx$$

$$= \frac{\pi x^7}{7}\Big|_0^1 = \frac{\pi\cdot 1^7}{7} - \frac{\pi\cdot 0^7}{7}$$

$$= \frac{\pi}{7}$$



10. (5 points) Let $p(t)$ denote the position of a particle in cm after t seconds. Suppose that we are unable to write down a nice formula for $p(t)$, but we happen to know that when $t = 2$ seconds, the particle has position 7 cm and velocity 30 cm/s. Find the linearization of $p(t)$ at $t = 2$ seconds, and use it to approximate the location of the particle when $t = 1.9$ seconds. (Include units with your answer.)

$$L(t) = p(2) + p'(2)(t-2) = 7 + 30(t-2)$$

$$p(1.9) \approx L(1.9) = 7 + 30(1.9-2) = 7 + 30\left(-\tfrac{1}{10}\right)$$

$$= 7 - 3 = 4 \text{ cm}$$

11. (6 points) Using the **limit definition of the derivative**, find $f'(3)$ if $f(x) = x^2 - 7$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{((3+h)^2 - \cancel{7}) - (3^2 - \cancel{7})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} (6+h)$$

$$= 6 + 0 = 6$$

12. (6 points) Suppose that a particle has position $s(t)$ feet at time t seconds and a velocity function $s'(t) = \sin^3(t)\cos(t)$ ft/s. Find the displacement (change in position) from time $t = 0$ seconds to time $t = \frac{\pi}{2}$ seconds. (Include units with your answer.)

$$s\left(\tfrac{\pi}{2}\right) - s(0) = \int_0^{\pi/2} \sin^3(t)\cos(t)\,dt$$

$u = \sin(t)$

$du = \cos(t)\,dt$

t	u
$\pi/2$	1
0	0

$$= \int_0^1 u^3\,du = \frac{u^4}{4}\Big|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4} \text{ ft}$$

13. (6 points) Find the absolute minimum and absolute maximum of $w(x) = (x-1)e^x$ on the interval $[-1, 1]$.

$w'(x) = 1 \cdot e^x + (x-1)e^x = x \cdot e^x$ is always defined.

$w'(x) = 0$ when $x = 0$.

$w(-1) = (-1-1)e^{-1} = \frac{-2}{e}$

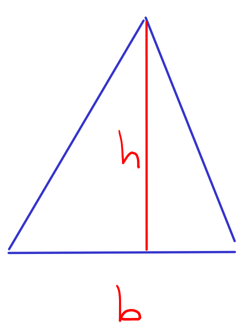
$w(0) = (0-1)e^0 = -1$ $\quad\left(-1 < \frac{-2}{e} < 0\right)$

$w(1) = (1-1)e^1 = 0$

On $[-1,1]$, $w(x)$ has an absolute max $w(1) = 0$

and an absolute min $w(0) = -1$.

14. (6 points) The base of a triangle is increasing at a rate of 2 ft/s, and the height of the triangle is increasing at a rate of 3 ft/s. Find the rate at which the area of the triangle is changing when the base is 4 ft and the height is 7 ft. (Include units with your answer.)



Want: $\frac{dA}{dt}$ when $b = 4$ ft and $h = 7$ ft

Know: $\frac{db}{dt} = 2$ ft/s and $\frac{dh}{dt} = 3$ ft/s

$A = \frac{1}{2}bh$

$\frac{dA}{dt} = \frac{1}{2}\frac{db}{dt} \cdot h + \frac{1}{2} \cdot b \cdot \frac{dh}{dt}$

when $b = 4$ ft and $h = 7$ ft,

$\frac{dA}{dt} = \frac{1}{2} \cdot 2 \cdot 7 + \frac{1}{2} \cdot 4 \cdot 3 = 7 + 6 = 13$ ft^2/s