Name Solutions Rec. Instr.

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Math 220

Final Exam

December 13, 2023

No books, calculators, or notes are allowed. *Please make sure that all cell phones, laptops, tablets, and smartwatches are turned off and put away.* You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		15	9		4
2		15	10		5
3		6	11		5
4		6	12		5
5		6	13		5
6		5	14		5
7		6	15		6
8		6	Total		100

1. (3 points each) Evaluate the following:

A. $\lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} = \lim_{x\to 5}(x+5) = 5+5 = 10$

B. $\lim_{x\to 0} \frac{e^x-1}{2x} \underset{\left(\substack{LH \\ \frac{0}{0}}\right)}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

C. $\int \left(\sqrt{x} - \frac{7}{x^2}\right)\, dx = \int (x^{1/2} - 7x^{-2})\,dx = \frac{2}{3}x^{3/2} - 7(-1)x^{-1} + C$

$= \frac{2}{3}x^{3/2} + \frac{7}{x} + C$

D. $\frac{d}{dx}\left(\frac{\sin(3x)}{x^{5/2}+1}\right) = \frac{\cos(3x)\cdot 3\cdot(x^{5/2}+1) - \sin(3x)\cdot\frac{5}{2}x^{3/2}}{(x^{5/2}+1)^2}$

E. $\frac{d}{d\theta}\left(\tan(\theta^3)\cdot\ln(5\theta)\right) = \sec^2(\theta^3)\cdot 3\theta^2 \ln(5\theta) + \tan(\theta^3)\cdot\frac{5}{5\theta}$

$= 3\theta^2\sec^2(\theta^3)\ln(5\theta) + \frac{\tan(\theta^3)}{\theta}$

2. (5 points each) Find the following:

A. $\int x\sqrt{x+1}\,dx$ $= \int (u-1)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right) du$

$u = x+1 \iff u-1 = x$

$\frac{du}{dx} = 1$

$du = dx$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

B. $\frac{dy}{dx}$ if $x^3 + y^3 = 5 - xy$

$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(5-xy)$

$3x^2 + 3y^2\frac{dy}{dx} = -y - x\frac{dy}{dx}$

$3y^2\frac{dy}{dx} + x\frac{dy}{dx} = -y - 3x^2$

$(3y^2+x)\frac{dy}{dx} = -y - 3x^2$

$\frac{dy}{dx} = \frac{-y-3x^2}{3y^2+x}$

C. $f'(x)$ if $f(x) = x^{2x}$

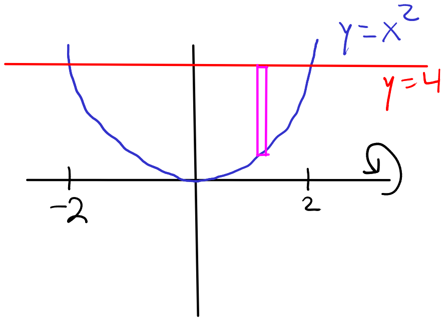
$\ln(f(x)) = \ln(x^{2x}) = 2x\ln(x)$

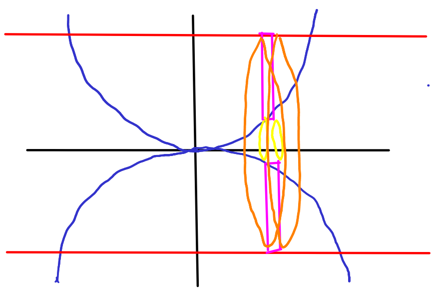
$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}(2x \cdot \ln(x))$

$\frac{f'(x)}{f(x)} = 2\ln(x) + 2x \cdot \frac{1}{x} = 2\ln(x) + 2$

$f'(x) = f(x)(2\ln(x)+2) = x^{2x}(2\ln(x)+2)$

3. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 4$ around the x-axis. (You do not need to simplify your final **numeric** answer.)





$$\text{Volume} = \int_{-2}^{2} \pi(4)^2\,dx - \int_{-2}^{2} \pi(x^2)^2\,dx$$

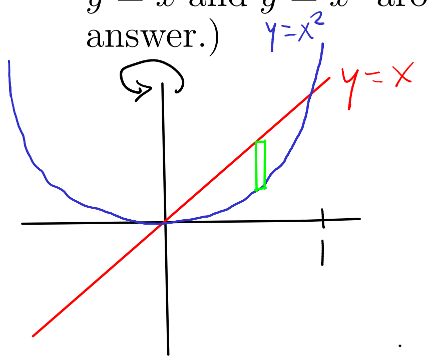
$$= \int_{-2}^{2} 16\pi\,dx - \int_{-2}^{2} \pi x^4\,dx$$

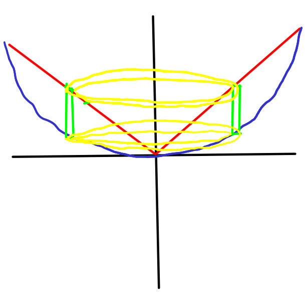
$$= (16\pi x)\Big|_{-2}^{2} - \left(\frac{\pi x^5}{5}\right)\Big|_{-2}^{2}$$

$$= (16\pi\cdot 2 - 16\pi(-2)) - \left(\frac{\pi\cdot 2^5}{5} - \frac{\pi(-2)^5}{5}\right)$$

$$= 64\pi - \frac{64\pi}{5} = \frac{256\pi}{5}$$

4. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ around the y-axis. (You do not need to simplify your final **numeric** answer.)





$$\text{Volume} = \int_{0}^{1} (2\pi x)(x - x^2)\,dx$$

$$= \int_{0}^{1} (2\pi x^2 - 2\pi x^3)\,dx$$

$$= \left(\frac{2\pi x^3}{3} - \frac{2\pi x^4}{4}\right)\Big|_{0}^{1}$$

$$= \left(\frac{2\pi\cdot 1^3}{3} - \frac{2\pi\cdot 1^4}{4}\right) - \left(\frac{2\pi\cdot 0^3}{3} - \frac{2\pi\cdot 0^4}{4}\right)$$

$$= \frac{2\pi}{3} - \frac{\pi}{2} = \frac{4\pi}{6} - \frac{3\pi}{6} = \frac{\pi}{6}$$

5. (6 points) Suppose that a particle has position $s(t)$ feet at time t seconds and a velocity function $s'(t) = \sin^4(t)\cos(t)$ ft/s. Find the displacement (change in position) from time $t = 0$ seconds to time $t = \frac{\pi}{2}$ seconds. (Include units with your answer.)

$$s\left(\tfrac{\pi}{2}\right) - s(0) = \int_0^{\frac{\pi}{2}} \sin^4(t)\cos(t)\,dt$$

$$= \int_0^1 u^4\,du$$

$u = \sin(t)$

$\frac{du}{dt} = \cos(t)$

$$= \frac{u^5}{5}\Big|_0^1$$

$du = \cos(t)\,dt$

$$= \frac{1^5}{5} - \frac{0^5}{5}$$

t	u
$\frac{\pi}{2}$	1
0	0

$$= \frac{1}{5}\text{ ft}$$

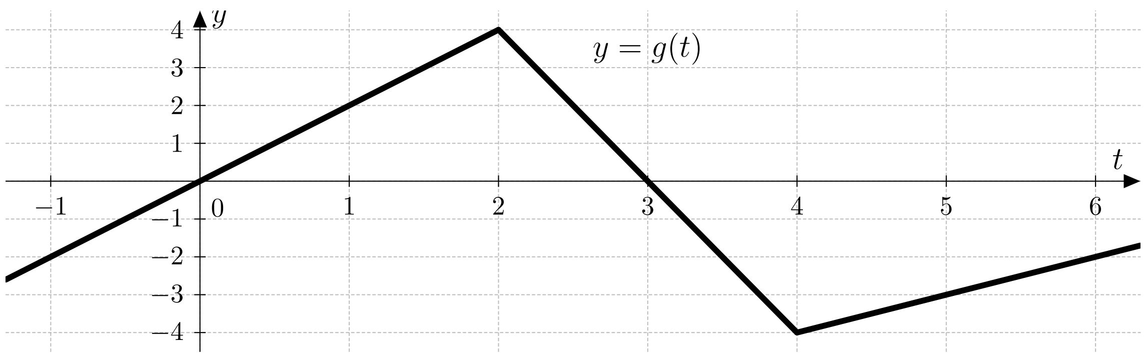
6. (5 points) Use a linearization of $u(x) = \sqrt{x}$ at $x = 9$ to approximate $\sqrt{9.6}$.

$u'(x) = \frac{1}{2\sqrt{x}}$ so the linearization of $u(x)$ at $x=9$ is

$L(x) = u(9) + u'(9)(x-9) = \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) = 3 + \frac{1}{6}(x-9).$

$\sqrt{9.6} = u(9.6) \approx 3 + \frac{1}{6}(9.6-9) = 3 + \frac{1}{6}(.6) = 3 + .1 = 3.1$

9.6 is close to 9

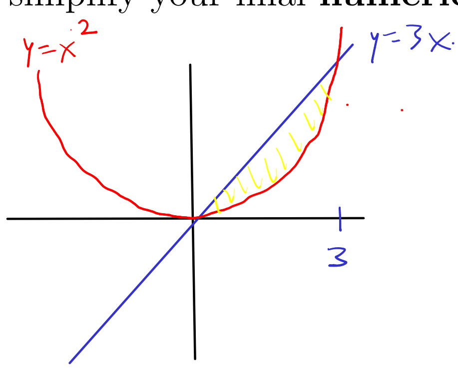


7. (3 points each) $y = g(t)$ is plotted above. Let $A(x) = \int_0^x g(t)\,dt$. Find the following quantities.

A. $A(3) =$ $\frac{1}{2} \cdot 3 \cdot 4 = 6$

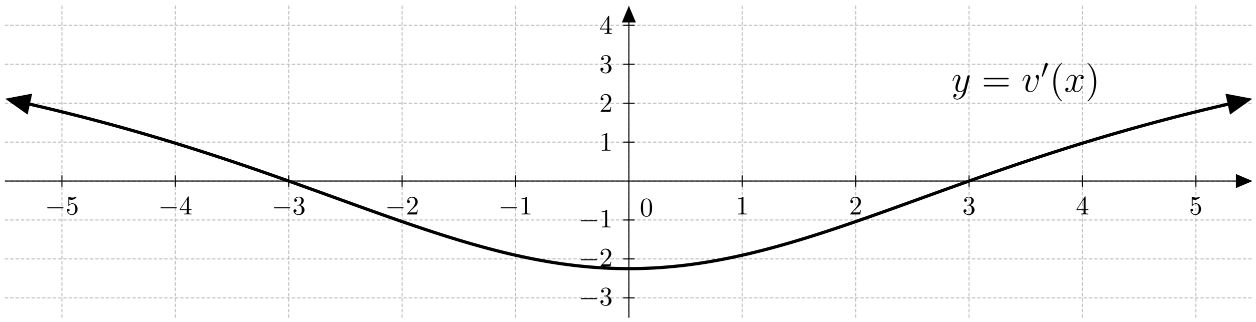
B. $A'(5) =$ $g(5) = -3$ because $A'(x) = \frac{d}{dx}\int_0^x g(t)\,dt = g(x)$

8. (6 points) Find the area between the curves $y = 3x$ and $y = x^2$. (You do not need to simplify your final **numeric** answer.)



$3x = x^2 \to 0 = x^2 - 3x = x(x-3) \to x = 0$ or 3

$$\text{Area} = \int_0^3 (3x - x^2)\,dx$$
$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3$$
$$= \left(\frac{3 \cdot 3^2}{2} - \frac{3^3}{3}\right) - \left(\frac{3 \cdot 0^2}{2} - \frac{0^3}{3}\right)$$
$$= \frac{27}{2} - 9$$
$$= \frac{27}{2} - \frac{18}{2}$$
$$= \frac{9}{2}$$



9. (2 points each) $y = v'(x)$ is plotted above. Find:

A. Interval(s) where $v(x)$ is increasing: $(-\infty,-3), (3,\infty)$ decreasing: $(-3,3)$

B. x-coordinate(s) where $v(x)$ has a local max: $x=-3$ local min: $x=3$

10. (5 points) Find $k(x)$ if $k'(x) = \sin(x) + x^{2/3}$ and $k(0) = 3$.

$k(x) = -\cos(x) + \frac{3}{5}x^{5/3} + C$ for some constant C.

$3 = k(0) = -\cos(0) + \frac{3}{5}\cdot 0^{5/3} + C = -1 + C$ so $C = 4$

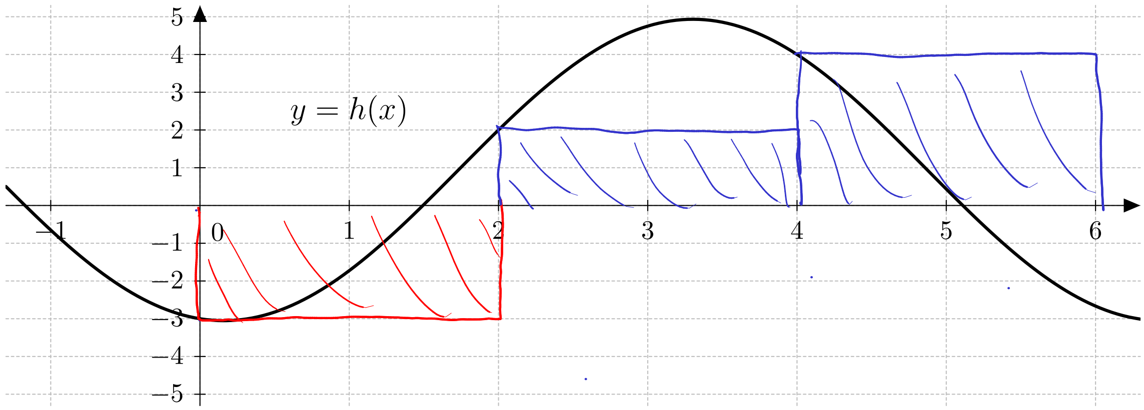
$k(x) = -\cos(x) + \frac{3}{5}x^{5/3} + 4$

11. (5 points) Using the **limit definition of the derivative**, find $q'(3)$ if $q(x) = 2x^2 - 7$.

$$q'(3) = \lim_{h\to 0} \frac{q(3+h) - q(3)}{h} = \lim_{h\to 0} \frac{(2(3+h)^2 - 7) - (2\cdot 3^2 - 7)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{18} + 12h + 2h^2 - \cancel{7} - \cancel{18} + \cancel{7}}{h} = \lim_{h\to 0} \frac{12h + 2h^2}{h}$$

$$= \lim_{h\to 0} (12 + 2h) = 12 + 2\cdot 0 = 12$$



12. (5 points) Estimate $\int_0^6 h(x)\,dx$ by computing L_3, the Left-Endpoint Approximation with 3 subintervals. Also, illustrate the corresponding rectangles on the graph above.

$$\Delta x = \frac{6-0}{3} = 2$$

$$\int_0^6 h(x)\,dx \approx L_3 = -3\cdot 2 + 2\cdot 2 + 4\cdot 2 = 6$$

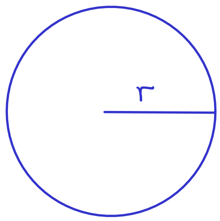
13. (5 points) Find the absolute minimum and maximum of $w(x) = x^3 - 3x^2 + 2$ on the interval $[1, 3]$.

$w'(x) = 3x^2 - 6x = 3x(x-2)$ is always defined.
$w'(x) = 0$ when $x = 0$ or 2. The only critical number in $[1,3]$ is $x = 2$.

$w(1) = 1^3 - 3\cdot 1^2 + 2 = 1 - 3 + 2 = 0$
$w(2) = 2^3 - 3\cdot 2^2 + 2 = 8 - 12 + 2 = -2$
$w(3) = 3^3 - 3\cdot 3^2 + 2 = 27 - 27 + 2 = 2$

On $[1,3]$, $w(x)$ has an absolute max $w(3) = 2$ and an absolute min $w(2) = -2$.

14. (5 points) Suppose that an oil spill from a ruptured tanker spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 ft/sec, how fast is the area of the spill increasing when the radius is 10 ft? (Include units with your answer.)



Want: $\frac{dA}{dt}$ when $r = 10$ ft

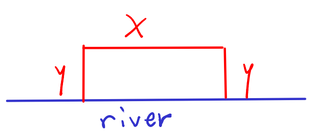
Know: $\frac{dr}{dt} = 2$ ft/s

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

when $r = 10$ ft, $\frac{dA}{dt} = 2\pi \cdot 10 \cdot 2 = 40\pi$ ft^2/s

15. (6 points) A farmer wants to fence off a 50 m^2 rectangular area that borders a straight river. The farmer needs no fencing along the river. What dimensions for the rectangle will minimize the length of fencing needed? (Make sure to justify why your answer corresponds to the absolute minimum, and include units with your answer.)



Minimize $\ell = x + 2y$

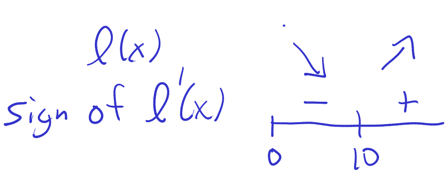
$50 = xy$ so $y = \frac{50}{x}$

Minimize $\ell(x) = x + 2\left(\frac{50}{x}\right) = x + \frac{100}{x}$ on $(0, \infty)$

$\ell'(x) = 1 - \frac{100}{x^2}$ is defined on $(0, \infty)$ and

$\ell'(x) = 0 \iff 1 - \frac{100}{x^2} = 0 \iff 1 = \frac{100}{x^2} \iff x^2 = 100$

$\iff x = \pm 10 \iff x = 10$ (x must be positive)



or $\ell''(x) = \frac{200}{x^3} > 0$ on $(0, \infty)$

$\ell(x)$ is minimized when $x = 10$ m and $y = \frac{50}{10} = 5$ m.