Name Solutions Rec. Instr.

Signature Rec. Time

Math 220
Exam 3
April 6, 2023

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 75 minutes to complete the exam. Unless instructed otherwise, **show your work** on each problem.

Problem	Points	Points Possible	Problem	Points	Points Possible
1		16	6		10
2		10	7		10
3		12	8		6
4		6	9		12
5		18	Total Score		100

1. The function $f(x)$ and its first and second derivatives are:

$$f(x) = \frac{x^2}{x^2+3} \qquad f'(x) = \frac{6x}{(x^2+3)^2} \qquad f''(x) = \frac{-18(x-1)(x+1)}{(x^2+3)^3}.$$

Find the information below about $f(x)$, and use it to sketch the graph of $f(x)$. When appropriate, write NONE. No work needs to be shown on this problem.

A. (1 point) Domain of $f(x)$: $(-\infty, \infty)$

B. (1 point) y-intercept: $f(0)=0$ so $(0,0)$

C. (1 point) x-intercept(s): $\frac{x^2}{x^2+3}=0$ when $x=0$ so $(0,0)$

D. (1 point) Horizontal asymptote(s): $y=1$

$\lim_{x\to-\infty} \frac{x^2}{x^2+3} = 1$ $\qquad$ $\lim_{x\to\infty} \frac{x^2}{x^2+3} = 1$

E. (1 point) Interval(s) $f(x)$ is increasing: $(0,\infty)$

F. (1 point) Interval(s) $f(x)$ is decreasing: $(-\infty, 0)$

$f'(x)$ is always defined and continuous.

$f'(x)=0$ when $x=0$

$f(x)$ ↘ ↗

sign of $f'(x)$ $-$ | $+$

0

local min

G. (1 point) Local maximum(s) (x, y): none

H. (1 point) Local minimum(s) (x, y): $(0,0)$

I. (1 point) Interval(s) $f(x)$ is concave up: $(-1,1)$

J. (1 point) Interval(s) $f(x)$ is concave down: $(-\infty,-1)$ and $(1,\infty)$

$f''(x)$ is always defined and continuous

$f''(x)=0$ when $x=\pm1$

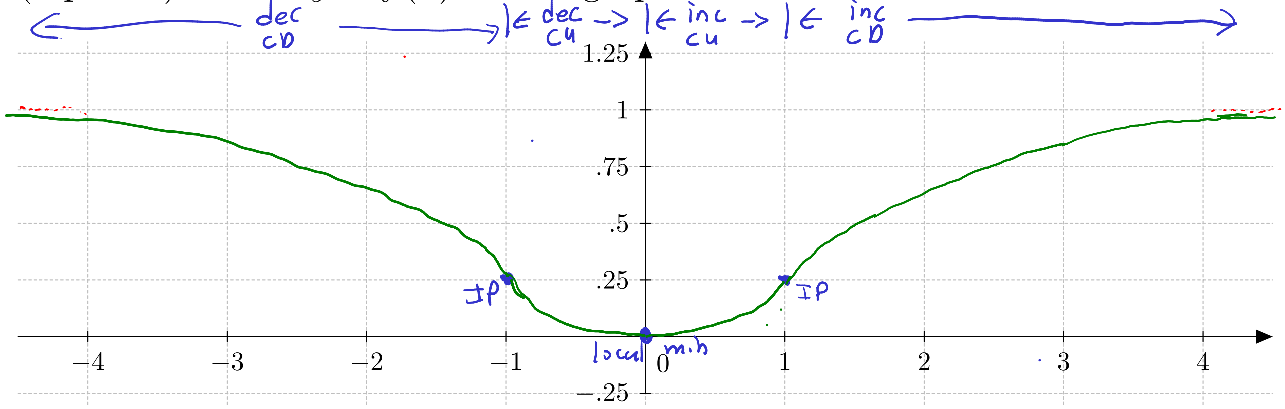
$f(x)$ CD CU CD

sign of $f''(x)$ $-$ | $+$ | $-$

-1 IP $\quad$ 1 IP

K. (1 point) Inflection point(s) (x, y): $(-1, \frac{1}{4})$ and $(1, \frac{1}{4})$

L. (5 points) Sketch $y = f(x)$ on the graph below.



2. (10 points) Find the absolute maximum and absolute minimum of $w(x) = x^3 - 3x + 2$ on $[0, 2]$.

$w'(x) = 3x^2 - 3 = 3(x-1)(x+1)$ is always defined.

$w'(x) = 0$ when $x = \pm 1$.

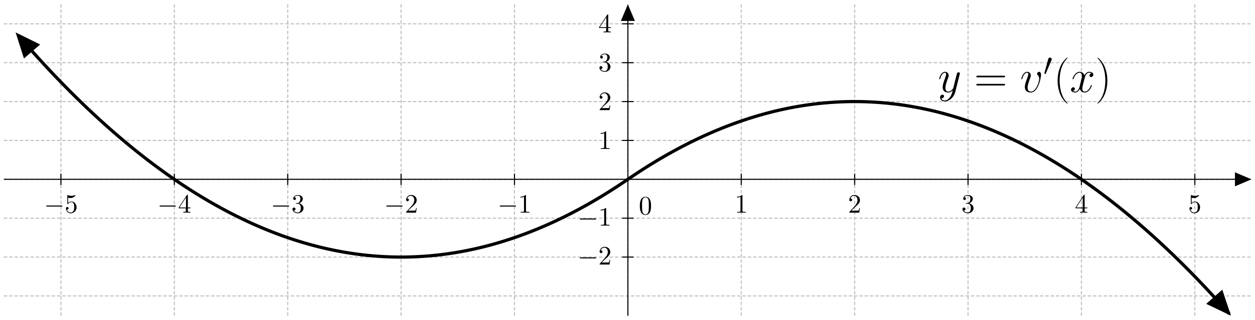
The only critical number in $[0,2]$ is $x=1$.

$w(0) = 0^3 - 3 \cdot 0 + 2 = 2$

$w(1) = 1^3 - 3 \cdot 1 + 2 = 0$

$w(2) = 2^3 - 3 \cdot 2 + 2 = 4$

On $[0,2]$, $w(x)$ has an absolute max at $(2,4)$ and an absolute min at $(1,0)$.



3. (3 points each) $y = v'(x)$ is plotted above. Find:

A. Interval(s) where $v(x)$ is increasing: $(-\infty, -4), (0, 4)$ decreasing: $(-4, 0), (4, \infty)$

B. x-coordinate(s) where $v(x)$ has a local max: $-4, 4$ local min: 0

C. Interval(s) where $v(x)$ is concave up: $(-2, 2)$ concave down: $(-\infty, -2), (2, \infty)$

D. x-coordinate(s) where $v(x)$ has an inflection point: $-2, 2$

4. (3 points each) In each of the following blanks, fill in "**max**" or "**min**".

A. If $h'(e) = 0$ and $h''(e) = 8$, then $h(x)$ has a local __min______ at $x = e$.

B. If $h'(-7) = 0$ and $h''(-7) = -5$, then $h(x)$ has a local __max______ at $x = -7$.

5. (6 points each) Find the following limits. (Use limit notation correctly.)

A. $$\lim_{x\to\infty} \frac{3+5x}{e^x+7} \underset{\left(\substack{LH \\ \frac{\infty}{\infty}}\right)}{=} \lim_{x\to\infty} \frac{5}{e^x} = 0$$

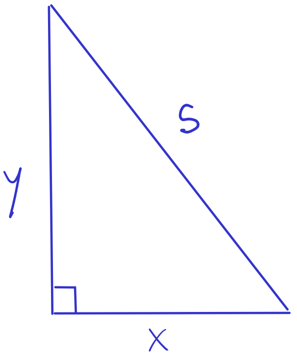
B. $$\lim_{\theta\to\pi} \frac{\sin(\theta)}{\theta-\pi} \underset{\left(\substack{LH \\ \frac{0}{0}}\right)}{=} \lim_{\theta\to\pi} \frac{\cos(\theta)}{1} = \cos(\pi) = -1$$

C. $$\lim_{x\to\infty} \frac{\sqrt{4x^2+3x}}{6x+7} = \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2+3x}}{x}}{\frac{6x+7}{x}} \underset{\substack{\uparrow \\ \boxed{\substack{x>0 \\ \text{so } x=\sqrt{x^2}}}}}{=} \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2+3x}}{\sqrt{x^2}}}{6+\frac{7}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{4x^2+3x}{x^2}}}{6+\frac{7}{x}} = \lim_{x\to\infty} \frac{\sqrt{4+\frac{3}{x}}}{6+\frac{7}{x}}$$

$$= \frac{\sqrt{4+0}}{6+0} = \frac{2}{3}$$

6. (10 points) A 5 meter ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of $\frac{1}{2}$ m/s, how fast will the bottom of the ladder be moving along the ground when the top of the ladder is 4 meters above the ground? (Include units with your answer.)



Want: $\frac{dx}{dt}$ when $y=4$ m

Know: $\frac{dy}{dt}=-\frac{1}{2}$ m/s

$x^2+y^2=5^2$

$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}5^2$

$2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=0$

$2x\frac{dx}{dt}=-2y\cdot\frac{dy}{dt}$

$\frac{dx}{dt}=\frac{-2y}{2x}\cdot\frac{dy}{dt}=-\frac{y}{x}\cdot\frac{dy}{dt}$

when $y=4$m, $4^2+x^2=5^2 \Rightarrow x^2=25-16=9 \Rightarrow x=\pm3 \Rightarrow x=3$m.

so $\frac{dx}{dt}=-\frac{4}{3}\cdot(-\frac{1}{2})=\frac{2}{3}$ m/s.

7. A. (7 points) Find the linearization of $g(x)=\ln(x)$ at $x=1$.

$g'(x)=\frac{1}{x}$ so the linearization of $g(x)$ at $x=1$ is

$L(x)=g(1)+g'(1)(x-1)=\ln(1)+\frac{1}{1}(x-1)=x-1$

B. (3 points) Use your answer from Part **A** to estimate $\ln(1.07)$.

$\ln(1.07)=g(1.07)\underset{\uparrow}{\approx}L(1.07)=1.07-1=.07$

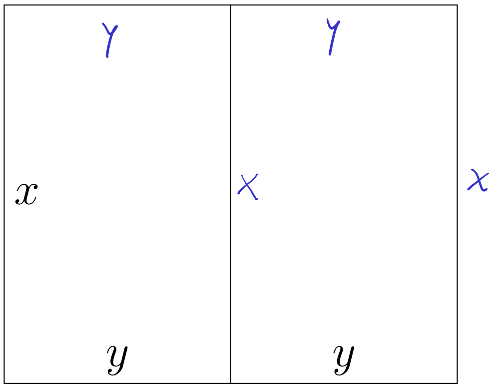
1.07 is close to 1

8. (6 points) The surface area A of a sphere of radius r is given by $A = 4\pi r^2$. Find the differential dA.

$$\frac{dA}{dr} = 8\pi r$$

$$dA = 8\pi r \cdot dr$$

9. (12 points) Suppose that you have 24 meters of fencing to make two adjacent rectangular kennels of length x meters and width y meters (see the diagram below). Find the values of x and y that maximize the enclosed area. (Justify why your answer corresponds to an absolute maximum, and include units in your answer.)



Maximize area $A = 2xy$.

$$3x + 4y = 24 \text{ m}$$

$$4y = 24 - 3x$$

$$y = \frac{24-3x}{4} = 6 - \frac{3}{4}x$$

$$A(x) = 2xy = 2x\left(6 - \frac{3}{4}x\right) = 12x - \frac{3}{2}x^2$$

$A'(x) = 12 - 3x$ is always defined.

$$A'(x) = 0 \Leftrightarrow 12 - 3x = 0 \Leftrightarrow 12 = 3x \Leftrightarrow x = 4 \text{ m}$$

$A(x)$ ↗ ↘

sign of $A'(x)$ + | −

0 4

$\boxed{\text{or}}$

$A''(x) = -3 < 0$ for all x

$\boxed{\text{or}}$

x and y must both be non-negative so maximize $A(x)$ on $[0, 8]$

The only critical number in $[0, 8]$ is $x = 4$.

$$A(0) = 12 \cdot 0 - \frac{3}{2} \cdot 0^2 = 0 \text{ m}^2$$

$$A(4) = 12 \cdot 4 - \frac{3}{2} \cdot 4^2 = 24 \text{ m}^2$$

$$A(8) = 12 \cdot 8 - \frac{3}{2} \cdot 8^2 = 0 \text{ m}^2$$

The area is maximized when $x = 4$m and $y = 6 - \frac{3}{4} \cdot 4 = 3$ m.