Name Solutions Rec. Instr.

Signature Rec. Time

Math 220

Final Exam

May 10, 2023

6:20-8:10 PM

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam. Unless instructed otherwise, **show your work.**

Problem	Points	Points Possible	Problem	Points	Points Possible
1		15	8		6
2		18	9		6
3		5	10		5
4		5	11		6
5		6	12		6
6		4	13		6
7		6	14		6

Total Score:

1. (3 points each) Evaluate the following. You do not need to simplify your final answers.

A. $\displaystyle\lim_{x\to\infty} \frac{-3x - 9e^x}{7x + 2e^x} \underset{\left(\substack{LH\\ \frac{-\infty}{\infty}}\right)}{=} \lim_{x\to\infty} \frac{-3-9e^x}{7+2e^x} \underset{\left(\substack{LH\\ \frac{-\infty}{\infty}}\right)}{=} \lim_{x\to\infty} \frac{-9e^x}{2e^x} = \lim_{x\to\infty} -\frac{9}{2} = -\frac{9}{2}$

B. $\displaystyle\int \left(\frac{2}{t^3} + 2\sqrt{t}\right) dt = \int \left(2t^{-3} + 2t^{1/2}\right) dt = 2\cdot\left(\frac{1}{-2}\right)t^{-2} + 2\cdot\left(\frac{2}{3}\right)t^{3/2} + C$

$\displaystyle = -t^{-2} + \frac{4}{3}t^{3/2} + C$

C. $\displaystyle\frac{d}{dx}\int_x^2 \cos(e^t)\,dt = \frac{d}{dx}\left(-\int_2^x \cos(e^t)\,dt\right) = -\cos(e^x)$

D. $\displaystyle\frac{d}{dx}\left(\frac{\tan(x^2)}{\cos(x) + x^3}\right) = \frac{\sec^2(x^2)\cdot 2x\left(\cos(x)+x^3\right) - \tan(x^2)\left(-\sin(x)+3x^2\right)}{\left(\cos(x)+x^3\right)^2}$

E. $\displaystyle\frac{d}{dx}\left(\ln(x)\cdot\arctan(x^3)\right) = \frac{1}{x}\cdot\arctan(x^3) + \ln(x)\cdot\frac{1}{1+(x^3)^2}\cdot 3x^2$

$\displaystyle = \frac{\arctan(x^3)}{x} + \frac{3\cdot x^2\ln(x)}{1+x^6}$

2. (6 points each) Find the following:

A. $\int_3^4 x\sqrt{x-3}\,dx$ $= \int_0^1 (u+3)\sqrt{u}\,du = \int_0^1 \left(u^{3/2} + 3u^{1/2}\right) du$

$= \left(\frac{2}{5}u^{5/2} + 3\cdot\left(\frac{2}{3}\right)u^{3/2}\right)\Big|_0^1$

$= \left(\frac{2}{5}u^{5/2} + 2u^{3/2}\right)\Big|_0^1$

$= \left(\frac{2}{5}\cdot 1^{5/2} + 2\cdot 1^{3/2}\right) - \left(\frac{2}{5}\cdot 0^{5/2} + 2\cdot 0^{3/2}\right)$

$= \frac{2}{5} + 2 = \frac{12}{5}$

$u = x - 3$ so $u + 3 = x$

$\frac{du}{dx} = 1$

$du = dx$

x	u
4	1
3	0

B. $\frac{dy}{dx}$ if $x^3 + y^3 = 5 - xy$

$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(5 - xy)$

$3x^2 + 3y^2\frac{dy}{dx} = -y - x\frac{dy}{dx}$

$3y^2\frac{dy}{dx} + x\frac{dy}{dx} = -y - 3x^2$

$(3y^2 + x)\frac{dy}{dx} = -y - 3x^2$

$\frac{dy}{dx} = \frac{-y-3x^2}{3y^2+x}$

C. $k'(x)$ if $k(x) = x^{\sin(x)}$

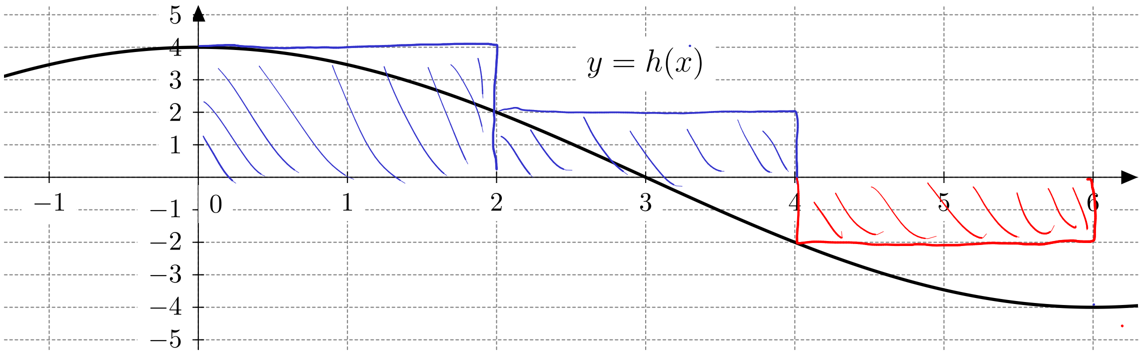
$\ln(k(x)) = \ln(x^{\sin(x)}) = \sin(x)\ln(x)$

$\frac{d}{dx}\ln(k(x)) = \frac{d}{dx}(\sin(x)\ln(x))$

$\frac{k'(x)}{k(x)} = \cos(x)\ln(x) + \sin(x)\cdot\frac{1}{x}$

$k'(x) = k(x)\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$

$= x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$



3. (5 points) $y = h(x)$ is plotted above. Estimate $\int_0^6 h(x)\,dx$ by using a Riemann sum with $n = 3$ subintervals, taking the sampling points to be left endpoints (the Left-Endpoint Approximation L_3). Also, illustrate the rectangles on the graph above.

$$\Delta x = \frac{6-0}{3} = 2$$

$$\int_0^6 h(x)\,dx \approx L_3 = 4\cdot 2 + 2\cdot 2 - 2\cdot 2 = 8$$

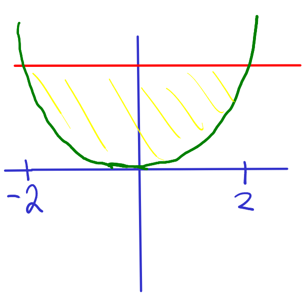
4. (5 points) Find $f(x)$ if $f'(x) = \cos(x) + 2$ and $f(0) = 5$.

$f(x) = \sin(x) + 2x + C$ for some constant C.

$5 = f(0) = \sin(0) + 2\cdot 0 + C = C$ so

$f(x) = \sin(x) + 2x + 5$

5. (6 points) Find the area between the curves $y = 4$ and $y = x^2$.

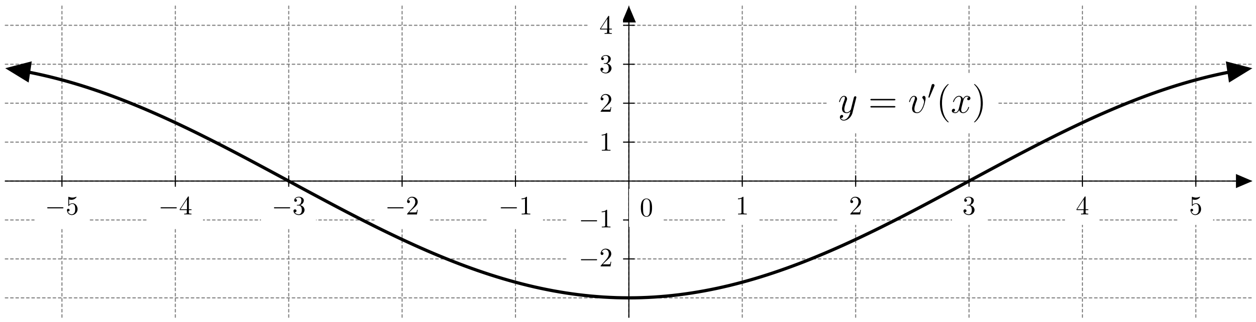


$$\text{Area} = \int_{-2}^{2} |4 - x^2|\,dx = \int_{-2}^{2} (4 - x^2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \left(4\cdot 2 - \frac{2^3}{3}\right) - \left(4\cdot(-2) - \frac{(-2)^3}{3}\right)$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3}$$

$$= 16 - \frac{16}{3} = \frac{48}{3} - \frac{16}{3} = \frac{32}{3}$$



6. (1 point each) $y = v'(x)$ is plotted above. Find:

A. Interval(s) where $v(x)$ is increasing: $(-\infty,-3), (3,\infty)$ decreasing: $(-3,3)$

B. x-coordinate(s) where $v(x)$ has a local max: -3 local min: 3

C. Interval(s) where $v(x)$ is concave up: $(0,\infty)$ concave down: $(-\infty,0)$

D. x-coordinate(s) where $v(x)$ has an inflection point: 0

7. (6 points) Let $p(x) = 75 - x^2$ be the price in dollars per meal that a chef can charge if they sell x meals. Revenue is the total amount of money received from the sale of x meals. Find the meal price that will maximize revenue? (Make sure to justify why your answer corresponds to the absolute maximum.)

We want to maximize the revenue

$R(x) = x \cdot p(x) = x(75-x^2) = 75x - x^3$ on $[0,\infty)$.

$R'(x) = 75 - 3x^2$ is always defined, and

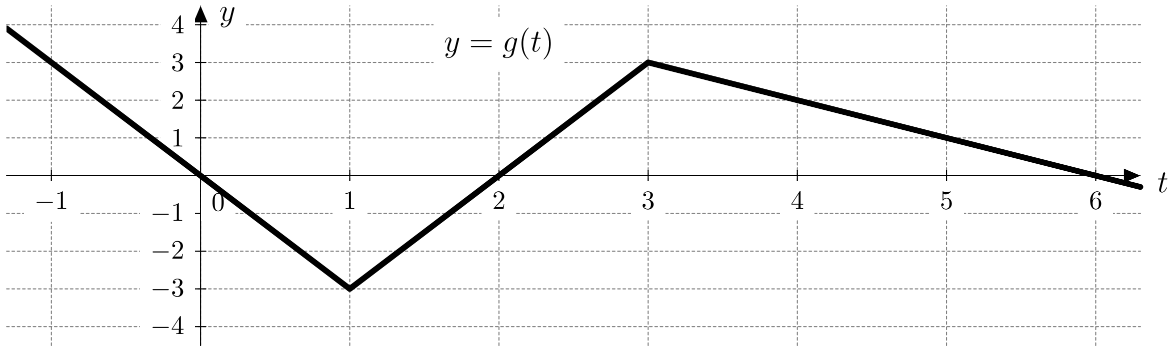
$R'(x) = 0 \iff 3x^2 = 75 \iff x^2 = 25 \iff x = \pm 5 \iff x = 5$

(x must be positive)

$R(x)$ $\nearrow$ $\searrow$

Sign of $R'(x)$ $+$ $-$ (0, 5) or $R''(x) = -6x < 0$ for $x > 0$

$R(x)$ is maximized when selling $x = 5$ meals at a price of $p(5) = 75 - 5^2 = \$50$ per meal.

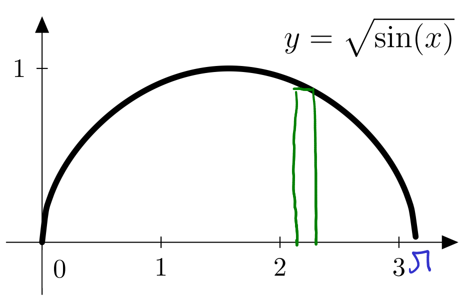


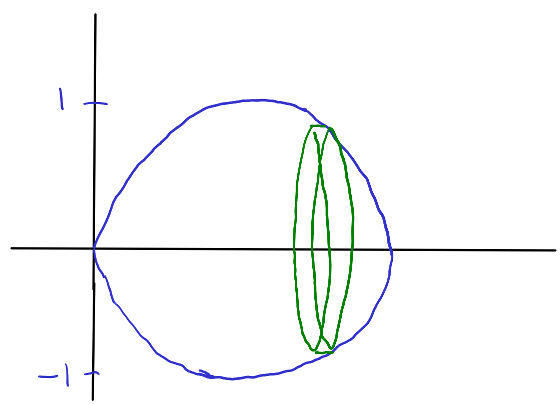
8. (3 points each) $y = g(t)$ is plotted above. Let $A(x) = \int_0^x g(t)\,dt$. Find the following quantities.

A. $A(3) =$ $\int_0^3 g(t)\,dt = -\frac{1}{2}\cdot 2\cdot 3 + \frac{1}{2}\cdot 1\cdot 3 = -3 + \frac{3}{2} = -\frac{3}{2}$

B. $A'(1) =$ $g(1) = -3$ because $A'(x) = \frac{d}{dx}\int_0^x g(t)\,dt = g(x)$

9. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{\sin(x)}$ and $y = 0$ between $x = 0$ and $x = \pi$ around the x-axis.





Volume $= \int_0^\pi \pi\left(\sqrt{\sin(x)}\right)^2 dx = \int_0^\pi \pi \sin(x)\,dx$

$= -\pi\cos(x)\Big|_0^\pi = -\pi\cos(\pi) + \pi\cos(0)$

$= \pi + \pi = 2\pi$

10. (5 points) Use the linearization of $u(x) = \sqrt{x}$ at $x = 25$ to approximate $\sqrt{24}$.

$u'(x) = \frac{1}{2\sqrt{x}}$. The linearization of $u(x)$ at $x=25$ is

$L(x) = u(25) + u'(25)(x-25) = \sqrt{25} + \frac{1}{2\sqrt{25}}(x-25) = 5 + \frac{1}{10}(x-25)$

$\sqrt{24} = u(24) \approx 5 + \frac{1}{10}(24-25) = 5 - \frac{1}{10} = 4.9$

24 is close to 25

11. (6 points) Using the **limit definition of the derivative**, find $f'(2)$ if $f(x) = x^2 + 5x$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\left((2+h)^2+5(2+h)\right)-(2^2+5\cdot 2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+4h+h^2+\cancel{10}+5h-\cancel{4}-\cancel{10}}{h} = \lim_{h\to 0} \frac{9h+h^2}{h}$$

$$= \lim_{h\to 0} (9+h) = 9+0 = 9$$

12. (6 points) Suppose that a particle has position $s(t)$ feet at time t seconds and a velocity function $s'(t) = t \cdot e^{-t^2}$ ft/s. Find the displacement (change in position) from time $t = 0$ seconds to time $t = 1$ second. (Include units with your answer.)

$$s(1) - s(0) = \int_0^1 t e^{-t^2}\, dt = \int_0^{-1} e^u \left(-\tfrac{1}{2}\right) du = -\tfrac{1}{2} e^u \Big|_0^{-1}$$

$u = -t^2$

$$= -\tfrac{1}{2}e^{-1} + \tfrac{1}{2}e^0 = \tfrac{1}{2} - \tfrac{1}{2e} \text{ ft}$$

$\frac{du}{dt} = -2t$

$\left(-\frac{1}{2}\right) du = t\, dt$

t	u
1	−1
0	0

13. (6 points) Find the absolute minimum and absolute maximum of $w(x) = x^3 - 3x + 2$ on the interval $[0, 2]$.

$w'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$ is always defined.

$w'(x) = 0$ when $x = \pm 1$

$x = 1$ is the only critical number in $[0,2]$

$w(0) = 0^3 - 3\cdot 0 + 2 = 2$

$w(1) = 1^3 - 3\cdot 1 + 2 = 0$

$w(2) = 2^3 - 3\cdot 2 + 2 = 4$

On $[0,2]$, $w(x)$ has an absolute max at $(2,4)$ and an absolute min at $(1,0)$.

14. (6 points) Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and the volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant, the volume is 40 cm^3, the pressure is 100 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume changing at this instant? (Include units with your answer.)

We want $\frac{dV}{dt}$ when $V = 40\text{ cm}^3$, $P = 100$ kPa, and $\frac{dP}{dt} = 20$ kPa/min.

$PV = C$

$\frac{d}{dt}[PV] = \frac{d}{dt} C$

$\frac{dP}{dt}\cdot V + P\cdot\frac{dV}{dt} = 0$

$P\cdot\frac{dV}{dt} = -\frac{dP}{dt}\cdot V$

$\frac{dV}{dt} = -\left(\frac{dP}{dt}\right)\cdot\left(\frac{V}{P}\right)$

At this instant, $\frac{dV}{dt} = -20\cdot\frac{40}{100} = \frac{-800}{100} = -8$ cm^3/min