Math 220 Sample Midterm 1

Name: ______________________________

Recitation instructor: ______________________________

Recitation time: ______________________________

- This is a closed-book, closed-notes exam. No calculators or electronic aids are permitted. Please make sure that your cell phone is turned off.
- Read each question carefully and show your work.
- You will have 75 minutes to complete the exam.

Problem 1. Evaluate the following limits. ① Direct substitution

A. $\lim_{x\to 2}(x^3+2x+1) = 2^3 + 2(2) + 1 = 13$

B. $\lim_{\theta\to\pi/2}\frac{\cos(\theta)}{\theta} = \frac{\cos\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{0}{\frac{\pi}{2}} = 0$

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{2}$

C. $\lim_{\theta\to 0}\frac{5(1-\cos\theta)}{\theta} = (-5)\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 5(0) = 0$

$\left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0\right]$

$\frac{5(1-\cos 0)}{0} = \frac{5(1-1)}{0} = \frac{0}{0}$

Problem 2.

$\lim_{x\to 0}\frac{1}{x} - \frac{1}{x(x-1)} \sim \frac{1}{0} - \frac{1}{0} \to$ common denominator

factor

A. $\lim_{t\to 1}\frac{t^2+t-2}{t-1}$

$\frac{0}{0}$

$= \lim_{t\to 1}\frac{(t+2)\cancel{(t-1)}}{\cancel{t-1}}$ $\to -2$ $\quad 2-1=1$

$= \lim_{t\to 1} t+2 = 1+2 = 3$

conjugate

B. $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$

$\frac{0}{0}$

$= \lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}$ $\quad (\sqrt{a})^2 = a$

$\boxed{a^2-b^2 = (a-b)(a+b)}$

$= \lim_{x\to 7}\frac{(\sqrt{x+2})^2 - 3^2}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7}\frac{x+2-9}{(x-7)(\sqrt{x+2}+3)}$

$= \lim_{x\to 7}\frac{\cancel{x-7}}{\cancel{(x-7)}(\sqrt{x+2}+3)} = \lim_{x\to 7}\frac{1}{\sqrt{x+2}+3} = \frac{1}{\sqrt{7+2}+3} = \frac{1}{6}$

Problem 3. Given that $\lim_{x\to 5} u(x) = 8$ and $\lim_{x\to 5} w(x) = 2$, find the following limits.

A. $\displaystyle\lim_{x\to 5} \frac{w(x)^2 - 9}{u(x)}$ $= \dfrac{2^2 - 9}{8} = -\dfrac{5}{8}$

not enough ✓

B. $\displaystyle\lim_{x\to 5} \frac{\sqrt{u(x)\cdot w(x)}}{x+5}$ $= \dfrac{\sqrt{8\cdot 2}}{5+5} = \boxed{\dfrac{\sqrt{16}}{10}} = \boxed{\dfrac{4}{10}} = \dfrac{2}{5}$ ✓

Problem 4. Use the squeeze theorem to find

$0 \sin\left(\frac{1}{0}\right)$ not good!

$$\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right)$$

$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$

$-|x| \le x \sin\left(\frac{1}{x^2}\right) \le |x|$

$f(x) \le g(x) \le h(x)$

$\lim_{x\to a} f(x) = L$

$\lim_{x\to a} h(x) = L$

$\Rightarrow \lim_{x\to a} g(x) = L$

$\lim_{x\to 0} |x| = |0| = 0$

$\lim_{x\to 0} -|x| = -|0| = 0$

equal

By squeeze thm, $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0$.

Problem 5. Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x \neq 1 \\ 6 & \text{if } x = 1. \end{cases}$$

Where is $f(x)$ continuous/discontinuous?

When $x \neq 1$, $f(x) = x^2+1$ is continuous.

When $x = 1$, we need to consider the followings.

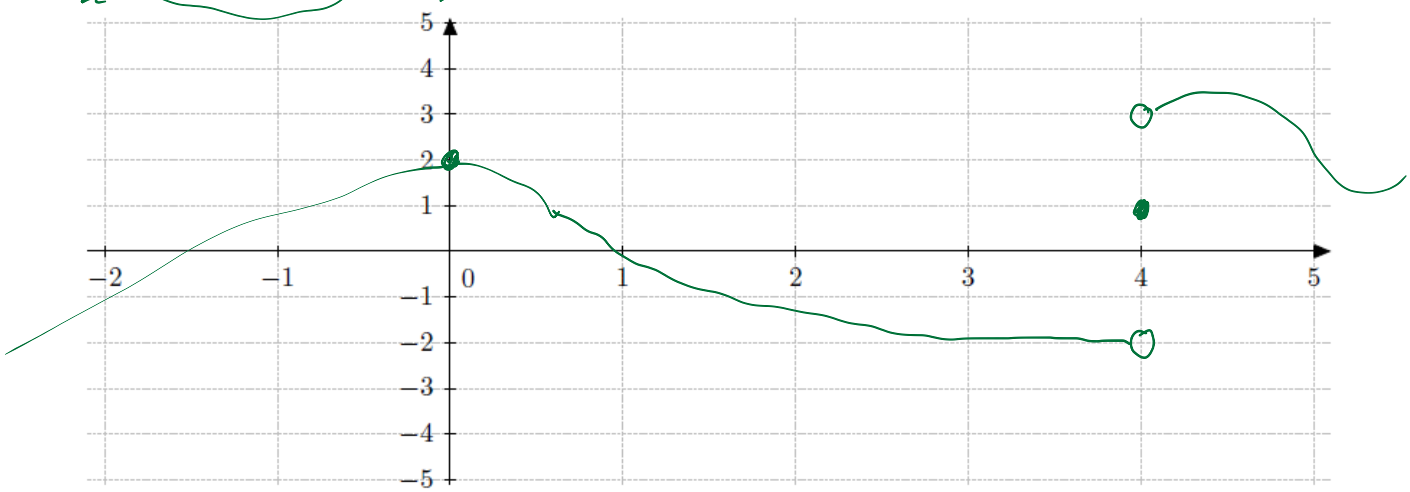
✓ ① $f(1) = 6$ is defined.

✓ ② $\lim_{x\to 1} f(x) = \lim_{x\to 1} x^2+1 = 1^2+1 = 2$ ($x \neq 1$)

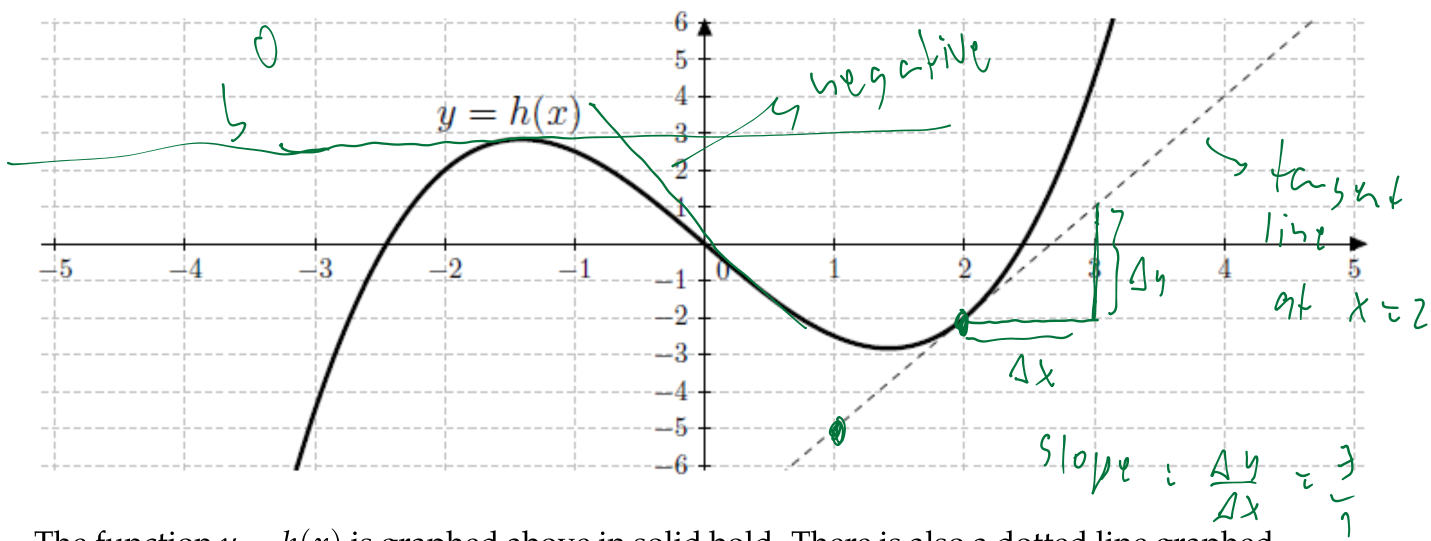
③ $\lim_{x\to 1} f(x) \neq f(1)$

$f(x)$ is discontinuous at $x=1$.

Problem 6. Sketch the graph of a function $k(x)$ that satisfies $\lim_{x\to 0} k(x) = 2$, $\lim_{x\to 4^-} k(x) = -2$, $\lim_{x\to 4^+} k(x) = 3$, and $k(4) = 1$.



Problem 7.



The function $y = h(x)$ is graphed above in solid bold. There is also a dotted line graphed. Find the following two values. [Answers are enough. No explanation is needed.]

A. $h(2) =$ -2

B. $h'(2) =$ 3

Problem 8. Suppose that an object is at position $s(t) = t^2 + 3$ feet at time t seconds.

t_1

A. Find the average velocity of the object over a time interval from time 3 seconds to time $3 + h$ seconds.

t_2

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(3+h) - s(3)}{3+h-3} = \frac{(3+h)^2 + 3 - (3^2+3)}{h}$$

$$= \frac{(3+h)^2 - 9}{h} \quad \text{ft/sec.}$$

B. Find the instantaneous velocity of the object at time 3 seconds by taking the limit of the average velocity in Part A as $h \to 0$.

$$\text{instantaneous velocity} \qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6+h)}{\cancel{h}} = \lim_{h \to 0} 6 + h = 6 + 0$$

$$= 6 \text{ ft/sec.}$$

Problem 9. Let $v(x) = \dfrac{2}{x}$.

A. Find $v'(1)$ by using one of the limit definitions of the derivative.

$$v'(1) = \lim_{y \to 1} \frac{v(y) - v(1)}{y - 1} = \lim_{y \to 1} \frac{\frac{2}{y} - \frac{2}{1}}{y - 1}$$

$$= \lim_{y \to 1} \frac{1}{y-1} \left[\frac{2 - 2y}{y} \right]$$

$$= \lim_{y \to 1} \frac{1}{\cancel{y-1}} \frac{2\,(\cancel{1-y})}{y} \to -(y-1)$$

$$= \lim_{y \to 1} \frac{-2}{y} = -\frac{2}{1} = \boxed{-2}$$

B. Find the equation of the tangent line to $y = v(x)$ at $x = 1$.

Line equation: $y - b = m(x - a)$

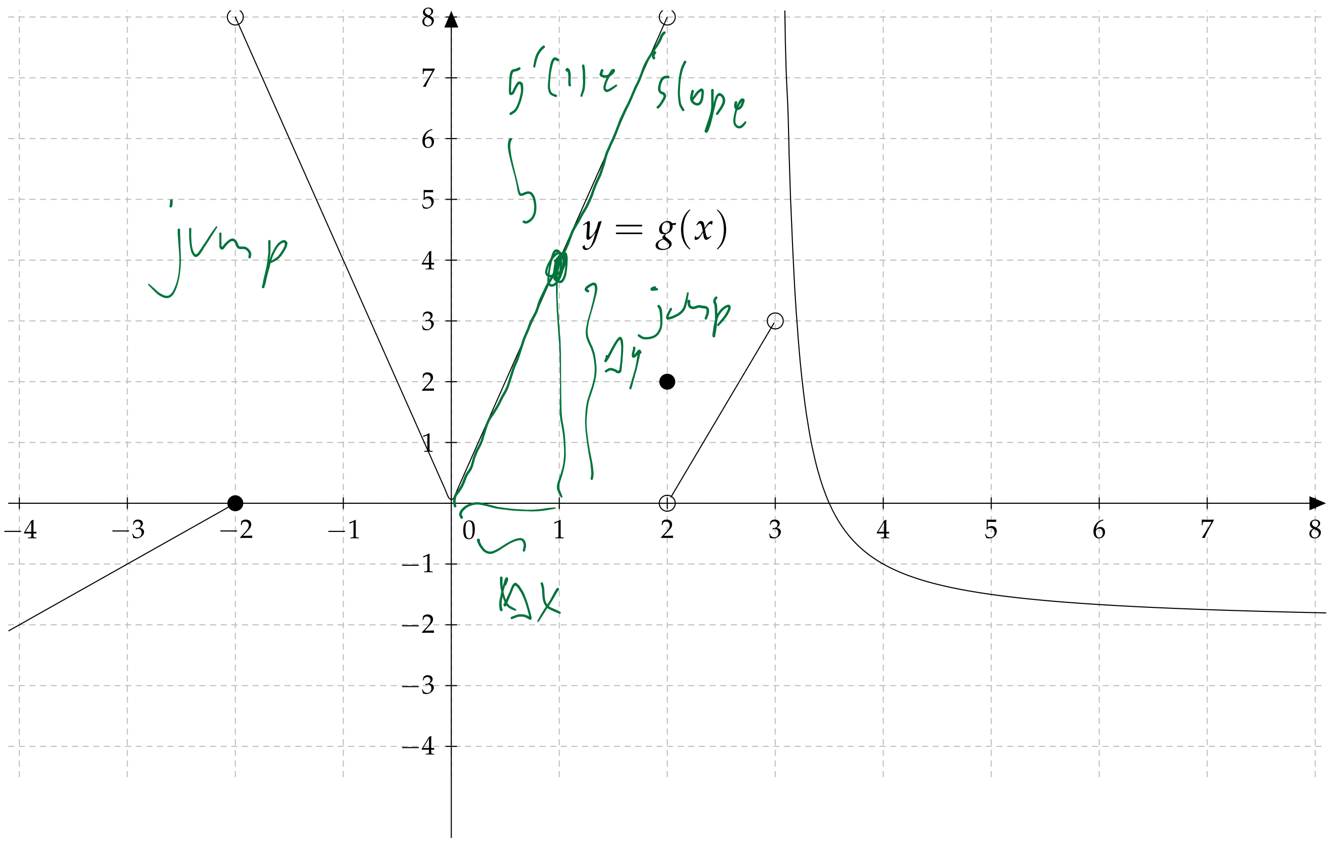
$m = v'(1) = -2$

$a = 1$, $b = v(1) = \frac{2}{1} = 2$

The tangent line equation is

$$\boxed{y - 2 = -2(x - 1)}$$

Problem 10.



State the value of each of the below quantities. If the quantity does not exist, write “does not exist” or ”DNE”. (Answers are enough. No explanation is needed.)

A. $\lim_{x\to -2^-} g(x) =$ 0

B. $\lim_{x\to -2^+} g(x) =$ 8

C. $\lim_{x\to -2} g(x) =$ DNE

D. $\lim_{x\to 3^+} g(x) =$ ∞

E. $\lim_{x\to 3^-} g(x) =$ 3

F. $g'(1) =$ 4

G. List all discontinuities and classify them as removable, infinite or jump

-2 jump

2 jump

3 infinite