Math 220 Sample Midterm 3

Name: Soln

Recitation instructor: ____________________

Recitation time: ____________________

- This is a closed-book, closed-notes exam. No calculators or electronic aids are permitted.
- Read each question carefully and show your work unless explicitly told otherwise.

Problem 1. Let $f(x) = x^4 - 2x^2$.

(a) (*points*) Find all critical numbers of $f(x)$ on the interval $(-\infty, \infty)$.

$f' = 4x^3 - 4x$

$= 4x(x^2 - 1)$

$f' = 0 \Rightarrow \boxed{x = 0, 1, -1}$

(b) (*points*) Find the absolute maximum and absolute minimum of $f(x)$ on $[-2, 2]$.

$f(-2) = 16 - 2 \cdot 4 = 8$

$f(-1) = 1 - 2 = -1$

$f(0) = 0$

$f(1) = 1 - 2 = -1$

$f(2) = 16 - 2 \cdot 4 = 8$

Abs max: 8

Abs min: -1

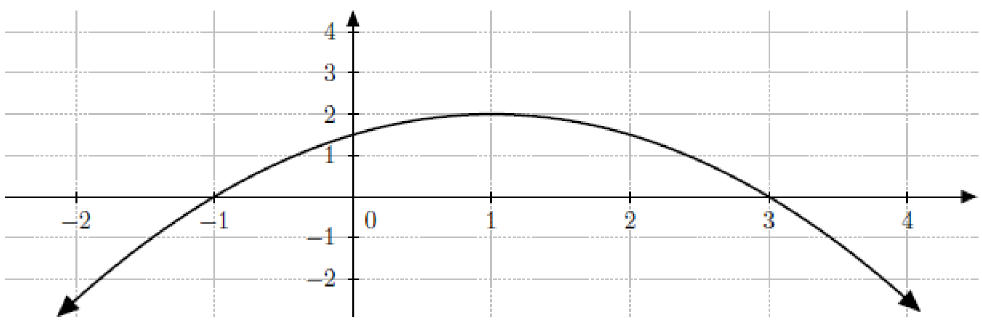
Problem 2. In each of the following blanks, fill in "**a local max**", "**a local min**" or "**neither** ".
(Note: "**neither**" means "neither a local max nor a local min") Also, no works need to be shown.

A. If $h'(3) = 0$ and $h''(3) = 2$, then $h(x)$ has local min at $x = 3$.

C.U.

B. If $h'(-2) = 0$ and $h''(-2) = -11$, then $h(x)$ has local max at $x = -2$.

C.D.



Above is the graph of $y = g'(x)$ to use for part C and D.

C. $g(x)$ has local min at $x = -1$.

D. $g(x)$ has neither at $x = 1$.

Problem 3. The function $f(x)$ and its first and second derivatives are:

$$f(x) = \frac{x^2-9}{x^2-4} \qquad f'(x) = \frac{10x}{(x^2-4)^2} \qquad f''(x) = -\frac{10(3x^2+4)}{(x^2-4)^3}.$$

Find the information below about $f(x)$, and use it to sketch the graph of $f(x)$. When appropriate, write NONE. No work needs to be shown on this problem.

A. (point) Domain of $f(x)$: $x \neq 2, -2$

B. (point) y-intercept: $(0, \frac{9}{4})$

C. (point) x-intercept(s): $(3,0)$, $(-3,0)$

D. (point) Horizontal asymptote(s): $y=1$

E. (point) Vertical asymptote(s): $x=2, -2$

$f' = \frac{10x}{(x^2-4)^2}$

f': $-$ | $-$ | $+$ | $+$ at -2, 0, 2

F. (point) Interval(s) $f(x)$ is increasing: $(0,2) \cup (2, \infty)$

G. (point) Interval(s) $f(x)$ is decreasing: $(-\infty, -2) \cup (-2, 0)$

H. (0.5 point) Local maximum(s) (x,y): None

I. (0.5 point) Local minimum(s) (x,y): $(0, \frac{9}{4})$

$$f'' = \frac{-10(3x^2+4)}{(x^2-4)^3} \qquad f'' \quad - \quad \underset{-2}{|} \quad + \quad \underset{2}{|} \quad -$$

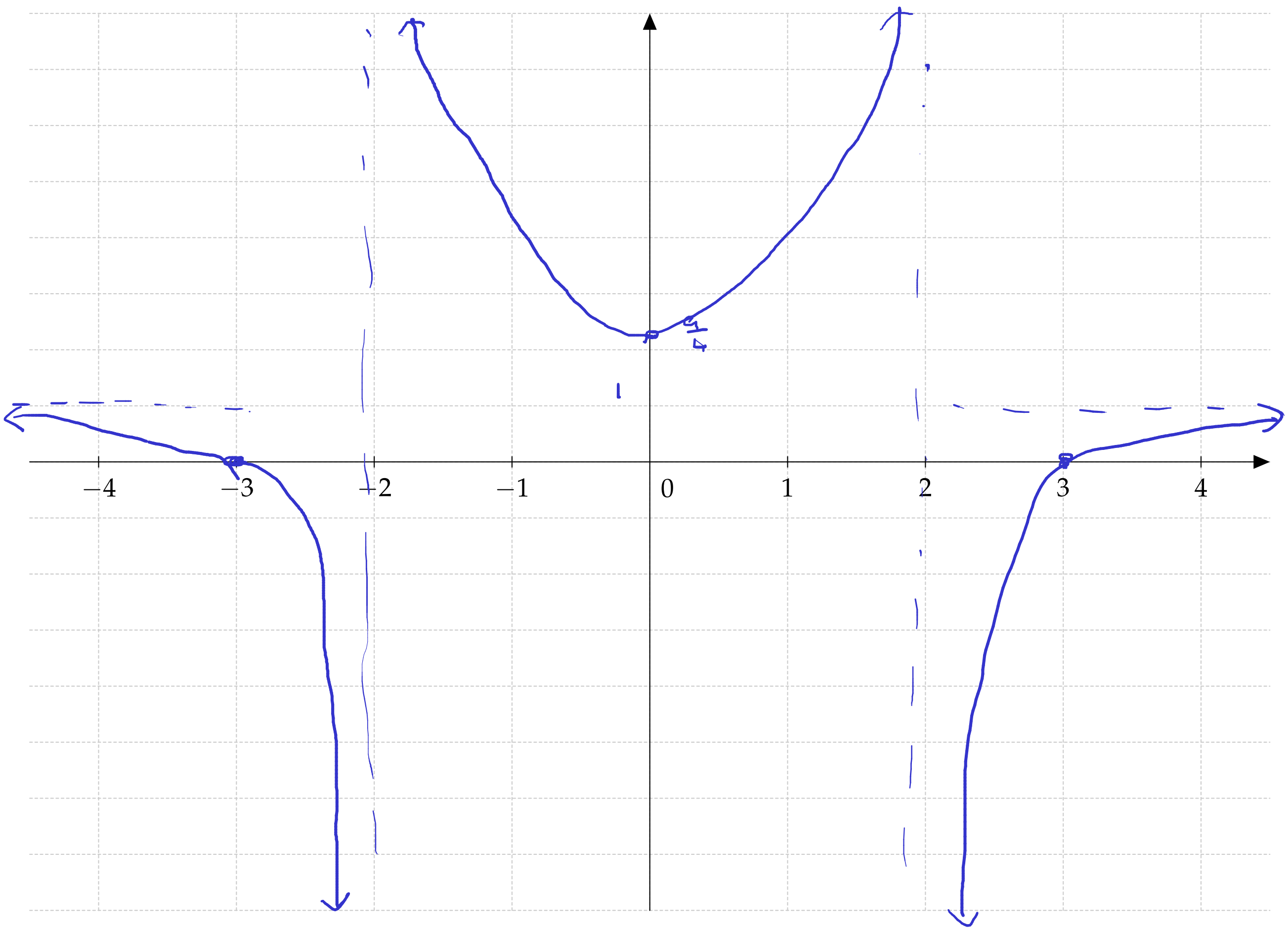
J. (point) Interval(s) $f(x)$ is concave up: $(-2, 2)$

K. (point) Interval(s) $f(x)$ is concave down: $(-\infty, -2) \cup (2, \infty)$

L. (point) Inflection point(s) (x, y): None

Both $x = 2, -2$ are V.A.

M. Sketch $y = f(x)$ on the graph below.



Problem 4. Find the following limits. (Use limit notation correctly.)

A. (*points*) $\lim_{x\to\infty} \dfrac{x^2-1}{e^x+2}$ $\quad \frac{\infty}{\infty}$

$\overset{LH}{=} \lim_{x\to\infty} \dfrac{2x}{e^x} \quad \frac{\infty}{\infty}$

$\overset{LH}{=} \lim_{x\to\infty} \dfrac{2}{e^x} = \boxed{0}$

B. (*points*) $\lim_{x\to 1^+} (x-1)^{x-1}$

$L =$

$\ln L = \lim_{x\to 1^+} (x-1)\ln(x-1)$

$= \lim_{x\to 1^+} \dfrac{\ln(x-1)}{(x-1)^{-1}} \quad \frac{\infty}{\infty}$

$\overset{LH}{=} \lim_{x\to 1^+} \dfrac{(x-1)^{-1}}{-(x-1)^{-2}} = -\lim_{x\to 1^+} x-1 = 0$

$\ln L = 0$

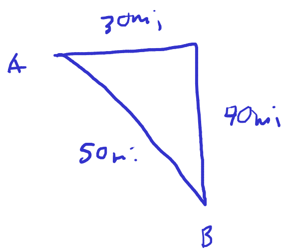
$\Rightarrow L = e^0 = \boxed{1}$

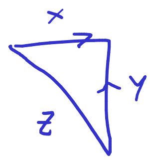
C. (*points*) $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+9}}{3x+2}$ $\quad \sim \dfrac{-x}{3x} = -\dfrac{1}{3}$

$= \lim_{x\to-\infty} \dfrac{\overset{-x=}{\sqrt{x^2}}\sqrt{1+\frac{9}{x^2}}}{x\left(3+\frac{2}{x}\right)}$

$= \lim_{x\to-\infty} \dfrac{-\sqrt{1+\cancel{\frac{9}{x^2}}}}{3+\cancel{\frac{2}{x}}} = \boxed{\dfrac{-1}{3}}$

Problem 5. Two airplanes are flying in the air at the same height: airplane A is flying east at 100 mi/h and airplane B is flying north at 200 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing? [**Include unit with your answer**]





$x' = -100$ mi/hr

$y' = -200$ mi/hr

Notice the signs!

$$\frac{d}{dt}\left(x^2+y^2 = z^2\right)$$

$$\cancel{2}xx' + \cancel{2}yy' = \cancel{2}zz'$$

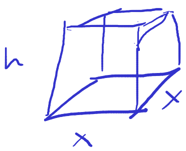
$$3\cancel{0}(-100) + 4\cancel{0}(-200) = 5\cancel{0}\,z'$$

$$-300 - 800 = 5z'$$

$$-1100 = 5z'$$

$$z' = \boxed{-220 \text{ mi/h}}$$

Problem 6.
If 12 ft^2 of material is available to make a box with square base and open top, find the largest possible volume for the box. **[Include unit with your answer]**



$V = x^2 h$

$A = x^2 + 4xh = 12$

$\Rightarrow x^2 - 12 = -4xh$

$\Rightarrow -\dfrac{x^2-12}{4x} = h$

$\Rightarrow h = -\frac{1}{4}x + \frac{3}{x}$

so

$V = x^2\left(-\frac{x}{4} + \frac{3}{x}\right)$

$= -\frac{1}{4}x^3 + 3x$

$V' = -\frac{3}{4}x^2 + 3 = 0$

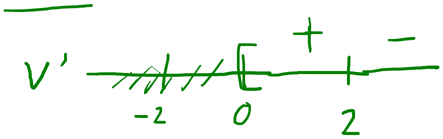
$\Rightarrow x^2 = 3 \cdot \frac{4}{3} = 4$

$\Rightarrow x = \pm 2$

$\Rightarrow x = 2$ ft $\leftarrow$ Claim this maximizes over $x \in (0,\infty)$

$\Rightarrow h = -\frac{1}{2} + \frac{3}{2} = 1$ ft

Justify:



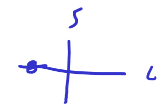
$\therefore$ abs max by 1st D.T.

$V'' = -\frac{3}{2}x < 0$ on $x \in (0,\infty)$

$\therefore$ abs max by 2nd D.T.

$V = x^2 h = 4 \cdot 1 = \boxed{4 \text{ ft}^3}$

Problem 7. Let $f(x) = \sin x$.



A. (*points*) Find the linearization for $f(x)$ at $x = \pi$.

$$L(x) = f(\pi) + f'(\pi)(x - \pi)$$
$$= 0 + -1\,(x - \pi)$$
$$L(x) = -(x - \pi)$$

$f(\pi) = 0$

$f'(x) = \cos x$

$f'(\pi) = -1$

B. (*points*) Use the linearization to approximate $f(\pi + 0.02)$.

$$f(\pi + 0.02) \approx L(\pi + 0.02) = \boxed{-0.02}$$

Problem 8. The area A of a circle with radius r is given by $A = \pi r^2$. Find the differential dA.

$$dA = 2\pi r\, dr$$