Math 220 Midterm 3

Name: Soln

Recitation instructor: ____________________

Recitation time: ____________________

- Exam 3 date/time: Thursday April 4, 2024 at 7:05 -8:20 pm.
- This is a closed-book, closed-notes exam. No calculators or electronic aids are permitted.
- Read each question carefully and show your work unless explicitly told otherwise.
- If you need extra room, use the blank page at the end of the exam. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Grading

1	/8	**6**	/10
2	/8	**7**	/8
3	/16	**8**	/3
4	/17		
5	/10	**Total**	/80

Problem 1. (*8 points*) Let $f(x) = x^3 - 3x + 1$.

(a) (*4 points*) Find all critical numbers of $f(x)$ on the interval $(-\infty, \infty)$.

$f' = 3x^2 - 3$

$= 3(x^2 - 1)$

$f' = 0 \implies \boxed{x = \pm 1}$

(b) (*4 points*) Find the absolute maximum and absolute minimum of $f(x)$ on $[0, 2]$.

$f(0) = 0 + 0 + 1 = 1$

$f(1) = 1 - 3 + 1 = -1$

$f(2) = 8 - 6 + 1 = 3$

Abs min : -1

Abs max : 3

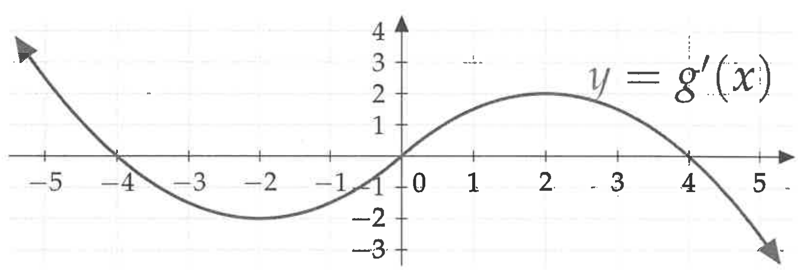
Problem 2. (*8 points*) (2 points each) In each of the following blanks, fill in "**a local max**", "**a local min**" or "**neither** ".
(Note: "**neither**" means "neither a local max nor a local min"). Also, no works need to be shown.

A. If $h'(3) = 0$ and $h''(3) = -2$, then $h(x)$ has local max at $x = 3$.

CD ∩

B. If $h'(-2) = 0$ and $h''(-2) = 3$, then $h(x)$ has local min at $x = -2$.

CU



Above is the graph of $y = g'(x)$ to use for part C and D.

C. $g(x)$ has Neither at $x = -2$.

D. $g(x)$ has local max at $x = 4$.

Problem 3. (*16 points*) The function $f(x)$ and its first and second derivatives are:

$$f(x) = \frac{x^2-4}{x^2-1} \qquad f'(x) = \frac{6x}{(x^2-1)^2} \qquad f''(x) = -\frac{6(3x^2+1)}{(x^2-1)^3}.$$

Find the information below about $f(x)$, and use it to sketch the graph of $f(x)$. When appropriate, write NONE. No work needs to be shown on this problem.

A. (1 point) Domain of $f(x)$: $x \neq -1, 1$ or $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

B. (1 point) y-intercept: $(0, 4)$

C. (1 point) x-intercept(s): $(2, 0)$, $(-2, 0)$

D. (1 point) Horizontal asymptote(s): $y = 1$

E. (1 point) Vertical asymptote(s): $x = 1, -1$

f': $-$ | $-$ | $+$ | $+$ at -1, 0, 1

F. (1 point) Interval(s) $f(x)$ is increasing: $(0, 1) \cup (1, \infty)$

G. (1 point) Interval(s) $f(x)$ is decreasing: $(-\infty, -1) \cup (-1, 0)$

H. (0.5 point) Local maximum(s) (x, y): None

I. (0.5 point) Local minimum(s) (x, y): $(0, 4)$

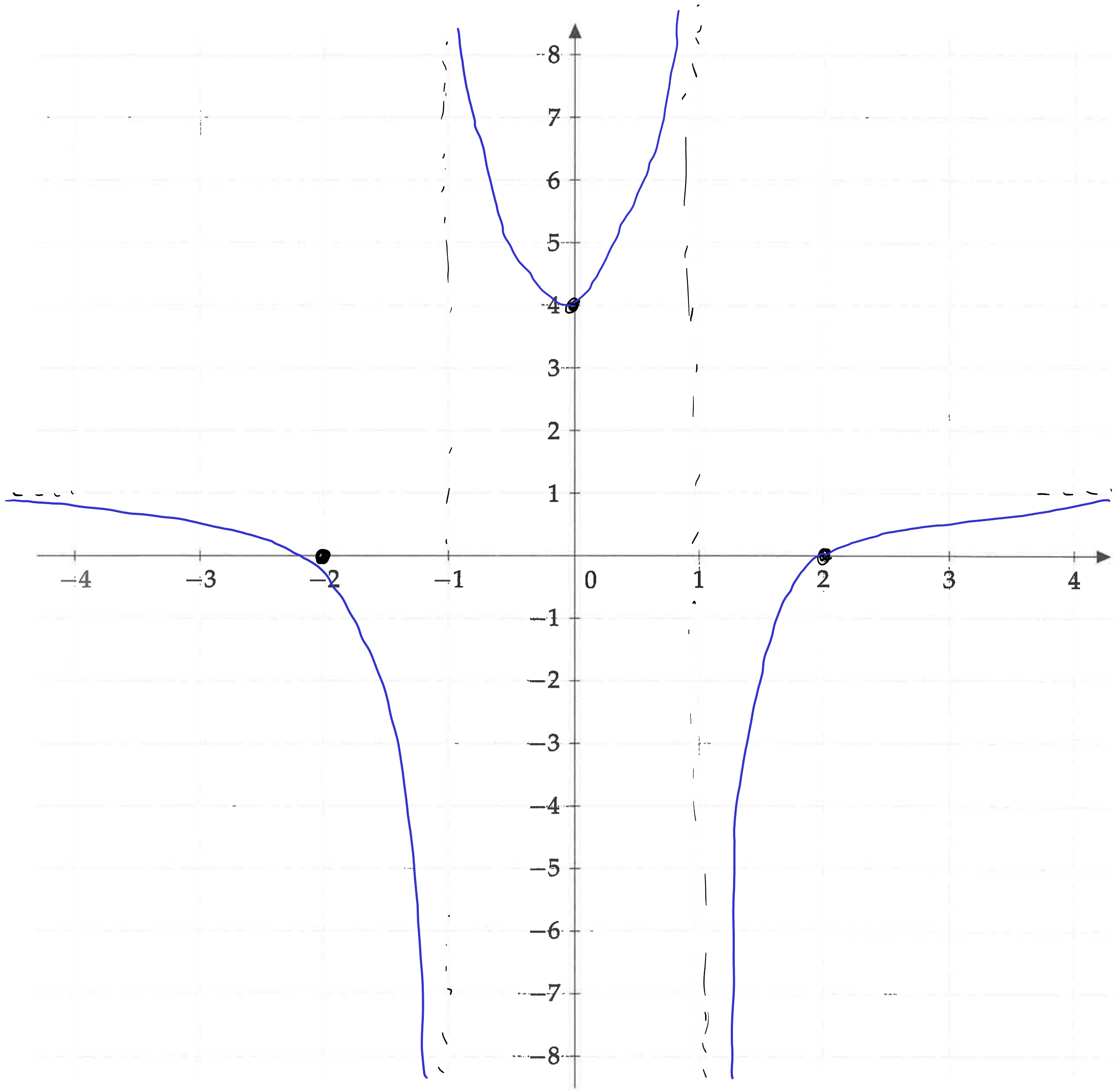
f'': $-$ | $+$ | $-$ at -1, 1

J. (1 point) Interval(s) $f(x)$ is concave up: $(-1, 1)$

K. (1 point) Interval(s) $f(x)$ is concave down: $(-\infty, -1) \cup (1, \infty)$

L. (1 point) Inflection point(s) (x, y): None $x = \pm 1$ are V.A.

M. (5 points) Sketch $y = f(x)$ on the graph below.



Problem 4. (*17 points*) Find the following limits. (Use limit notation correctly.)

A. (*5 points*) $\lim_{x\to\infty} \frac{x+e^x}{1-5x^2}$ $\overset{LH}{=} \lim_{x\to\infty} \frac{1+e^x}{-10x}$ $\overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{-10} = \boxed{-\infty}$

$\frac{\infty}{\infty}$ $\frac{\infty}{\infty}$

B. (*6 points*) $\lim_{x\to 0^+} x^x$ $= L$ $\quad 0^0$

$\ln L = \lim_{x\to 0^+} x \ln x \quad 0\cdot\infty$

$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \quad \frac{\infty}{\infty}$

$\overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0 \quad \therefore \ln L = 0$

$\Rightarrow L = e^0 = \boxed{1}$

C. (*6 points*) $\lim_{x\to-\infty} \frac{\sqrt{9x^2+16}}{x+10}$ $\sim \frac{3\sqrt{x^2}}{x} = \frac{-3x}{x} = -3$

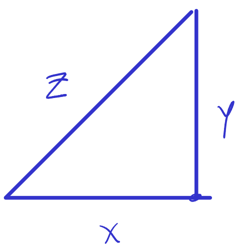
since $x<0$

$= \lim_{x\to-\infty} \frac{\overset{-x=}{\sqrt{x^2}}\sqrt{9+\frac{16}{x^2}}}{x\left(1+\frac{10}{x}\right)}$

$= \lim_{x\to-\infty} \frac{-\sqrt{9+\cancel{\frac{16}{x^2}}^{\to 0}}}{1+\cancel{\frac{10}{x}}^{\to 0}}$

$= \boxed{-3}$

Problem 5. (*10 points*) Two people start driving from the same point. One person travels west at a rate of 30 miles per hour, and the other drives north at a rate of 40 miles per hour. At what rate is the distance between the two people changing 1 hour after they start driving? **[Include unit with your answer]**

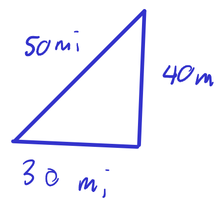


$x' = 30$ mph

$y' = 40$ mph

Find z' after 1 hr.

After 1 hr:



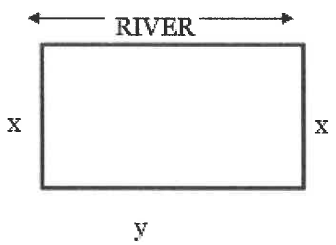
$$\frac{d}{dt} \quad x^2 + y^2 = z^2$$

$$\cancel{2}xx' + \cancel{2}yy' = \cancel{2}zz'$$

$$30(30) + 40(40) = 50z'$$

$$\Rightarrow \boxed{z' = 50 \text{ mph}}$$

Problem 6. (*10 points*) You have 400 ft of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area? (i.e. What are x and y (in the figure below) that maximizes the area?) Make sure to justify why your answer corresponds to an absolute maximum. **Include unit with your answer.**



$A = xy$

$400 = 2x + y$

$\Rightarrow y = 400 - 2x$

$A = x(400 - 2x)$

$A(x) = 400x - 2x^2$

$A'(x) = 400 - 4x = 0$

$\Rightarrow \boxed{x = 100 \text{ ft}}$ ← Claim this maximizes area

$y = 400 - 2(100)$

$\boxed{y = 200 \text{ ft}}$

1st D

A': $+$ | $-$ at 100

$\therefore$ $x = 100$ is Abs max

2nd D

$A'' = -4 < 0$

$\therefore$ Abs max @ $x = 100$

CIM

$A(x)$ has natural domain $x \in [0, 200]$

$A(0) = 0$

$A(200) = 0$

$A(100) = 100(400 - 200)$

$= 100 \cdot 200 > 0$

$\therefore$ $x = 100$ is Abs max

Problem 7. (*8 points*) Let $f(x) = x^{100}$.

A. (*5 points*) Find the linearization for $f(x)$ at $x = 1$.

$f(1) = 1$

$f'(x) = 100x^{99}$

$f'(1) = 100$

$L(x) = 1 + 100(x - 1)$

B. (*3 points*) Use the linearization to approximate $f(1.001)$.

$$f(1.001) \approx L(1.001)$$
$$= 1 + 100(.001)$$
$$= 1 + 0.1$$
$$= \boxed{1.1}$$

Problem 8. (*3 points*) The volume V of a sphere with radius r is given by $V = \frac{4}{3}\pi r^3$. Find the differential dV.

$$dV = \frac{4}{3}\pi \cdot 3r^2\, dr$$
$$= \boxed{4\pi r^2 \cdot dr}$$