Math 220 Sample Final Exam

Name: Soln

Recitation instructor: ______________________

Recitation time: ______________________

- This is a closed-book, closed-notes exam. No calculators or electronic aids are permitted.
- Read each question carefully and show your work unless explicitly told otherwise.

Problem 1. (*15 points*)
(3 points each) Evaluate the following:

A. Use L'Hopital rule to evaluate $\lim_{\theta \to 0} \frac{\cos(\theta^2) - 1}{\theta^2}$.

$\frac{1-1}{0}$

$\overset{LH}{=} \lim_{\theta \to 0} \frac{-\sin(\theta^2) \cdot \cancel{2\theta}}{\cancel{2\theta}} = 0$

B. $\int \left(x^{-1/5} + \cos(x) + \frac{1}{x} \right) dx =$

$\frac{5}{4} x^{\frac{4}{5}} + \sin x + \ln|x| + C$

C. $\frac{d}{dx} \int_{x^3}^{1} e^{\sin t}\, dt =$ $-\frac{d}{dx} \int_{1}^{x^3} e^{\sin t}\, dt$

$= - e^{\sin x^3} \cdot 3x^2$

D. $\frac{d}{dx}\left(\frac{\tan(x)}{e^x + \ln(x)}\right) =$ $\dfrac{(e^x + \ln x)\sec^2 x - \tan x\,(e^x + \frac{1}{x})}{(e^x + \ln x)^2}$

E. $\frac{d}{dx}\left(\sin(5^x) \cdot \arctan(x)\right) =$ $\cos(5^x) \cdot \ln 5 \cdot 5^x \cdot \arctan x$

$+ \sin(5^x) \cdot \dfrac{1}{1+x^2}$

Problem 2. (*5 points*) Using the limit definition of the derivative, find $f'(1)$ if $f(x) = x^2 - 5x$.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 5(1+h) - [1-5]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{5} - 5h \cancel{-1} + \cancel{5}}{h}$$

$$= \lim_{h \to 0} 2 + h - 5 = -3$$

Chek:

$f' = 2x - 5$

$f'(1) = 2 - 5 = -3$ ✓

Problem 3. (*10 points*)

A. (*5 points*) Use implicit differentiation to find $\frac{dy}{dx}$ for $x - 3x^2y + y = e^x$.

$$\xrightarrow{\frac{d}{dx}} \quad 1 - 3(2xy + x^2y') + y' = e^x$$

$$\Rightarrow \quad 1 - 6xy - 3x^2y' + y' = e^x$$

$$\Rightarrow \quad y'(-3x^2+1) = e^x - 1 + 6xy$$

$$\Rightarrow \quad \boxed{y' = \frac{e^x - 1 + 6xy}{-3x^2+1}}$$

B. (*5 points*) Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = x^{\sec(x)}$.

$$\xrightarrow{\ln(-)} \quad \ln y = \sec x \cdot \ln x$$

$$\xrightarrow{\frac{d}{dx}} \quad \frac{y'}{y} = \sec x \cdot \tan x \cdot \ln x + \frac{\sec x}{x}$$

$$\Rightarrow \quad \boxed{y' = x^{\sec x}\left(\sec x \cdot \tan x \cdot \ln x + \frac{\sec x}{x}\right)}$$

w'' $\quad - \quad | \quad +$
$\qquad\qquad 2$

Problem 4. (3 *points*) (1 point each) For the function $w(x)$, one has $w''(x) = \frac{x-2}{\sqrt{x^2+2}}$. ← always >0

Find the following:

A. Interval(s) where $w(x)$ is concave up: $(2, \infty)$

B. Interval(s) where $w(x)$ is concave down: $(-\infty, 2)$

C. x-coordinate(s) where $w(x)$ has an inflection point: $x = 2$

Problem 5. (6 *points*) (6 points) Use a linearization of $u(x) = 12x^{1/3}$ at $x = 8$ to approximate $12(8.1)^{1/3}$.

$$L(x) = u(8) + u'(8)(x-8)$$
$$= 24 + 1(x-8)$$

$u(8) = 12 \cdot 2 = 24$

$u'(x) = 4x^{-\frac{2}{3}}$

$u'(8) = 4 \cdot \frac{1}{4} = 1$

$$12(8.1)^{1/3} = u(8.1)$$
$$\approx L(8.1)$$
$$= 24 + (8.1 - 8)$$
$$= \boxed{24.1}$$

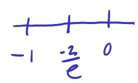
Problem 6. (*6 points*) Find the absolute minimum and maximum of $w(x) = (x-1)e^x$ on the interval $[-1, 1]$.

$$w' = 1 \cdot e^x + (x-1)e^x$$

$$= x e^x$$

$w' = 0 \implies x = 0 \leftarrow$ crit pt.

$$w(-1) = -2e^{-1} \approx \frac{-2}{2.7}$$

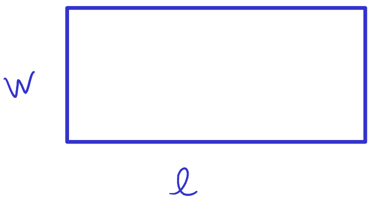


$$w(0) = -1e^0 = -1$$

$$w(1) = 0 \cdot e^1 = 0$$

$$\boxed{\begin{array}{l} \text{Abs max} : 0 \\ \text{Abs min} : -1 \end{array}}$$

Problem 7. (*8 points*) Suppose that the length of a rectangle is decreasing at a rate of 5 m/s and the width is increasing at a rate of 10 m/s. How fast is the area changing when the length is 10 m and the width is 30 m ? Is the area increasing or decreasing? (Include units with your answer.)



$\ell' = -5$ m/s

$w' = 10$ m/s

Find A' when $\ell = 10$ m, $w = 30$ m

$A = \ell w$

$\frac{d}{dt}$

$A' = \ell' w + \ell w'$

$= (-5)\,30 + 10 \cdot 10$

$= -150 + 100 = \boxed{-50\ \text{m}^2/\text{s} \quad \text{decreasing area}}$

Problem 8. (*7 points*) A car rental agency rents 200 cars per day at a rate of 30 dollars per day. For each 1 dollar increase in the daily rate, 5 fewer cars are rented. At what rate should the cars be rented to produce the maximum income? (Make sure to justify why your answer corresponds to the absolute maximum.)

200 cars/day

30 \$ /day

1\$ incr $\to$ 5 car decr

$C(p) = 200 - 5(p-30)$

cars rented, given price to rent

$= -5p + 200 + 150$

$= -5p + 350$

$I(p) = p \cdot C(p)$

income, given price to rent

$= -5p^2 + 350p$ $\leftarrow$ maximize this on $p \geq 30$

$I'(p) = -10p + 350 = 0$

$\Rightarrow 350 = 10p$

$\Rightarrow p = 35$ $\leftarrow$ $\boxed{\text{rent price of \$35 maximizes income}}$

Justification:

$I'' = -10 < 0$

By 2nd D.T. for A.E.V., $p = 35$ is an abs. max.

(other methods may work)

Problem 9. (*12 points*)
(6 points each) Evaluate the following:

A. $\int \frac{\ln(x)^{3/2}}{x}\,dx$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$= \int u^{\frac{3}{2}}\,du$

$= \frac{2}{5} u^{\frac{5}{2}}$

$= \frac{2}{5}\left[\ln(x)\right]^{\frac{5}{2}} + C$

B. $\int_0^{1/5^{1/6}} \frac{x^2}{\sqrt{1-\underbrace{4x^6}_{=u^2}}}\,dx$

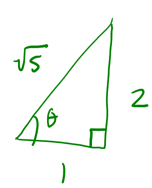
$u = 2x^3$
$du = 6x^2\,dx$

$x = 0 \to u = 0$
$x = \frac{1}{5^{1/6}} \to u = 2 \cdot \frac{1}{5^{\frac{1}{2}}}$

$= \frac{1}{6}\int_0^{\frac{2}{5^{1/2}}} \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{6}\sin^{-1}(u)\Big|_0^{\frac{2}{5^{1/2}}} = \frac{1}{6}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

Note that WolframAlpha gives Arctan(2)/6 which is equivalent to the above answer.
This can be checked numerically (both are roughly 0.1845247863) or by considering the triangle:



One can observe that

$$\theta = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = \tan^{-1}(2)$$

Problem 10. (*6 points*) Suppose that a particle has position $s(t)$ feet at time t seconds and a velocity function $s'(t) = \sin^3 t \cos t$ ft/s. Find the displacement (change in position) from time $t = 0$ seconds to time $t = \frac{\pi}{2}$ seconds. (Include units with your answer.)

$$\text{disp} = \int_0^{\pi/2} \sin^3 t \cos t \, dt$$

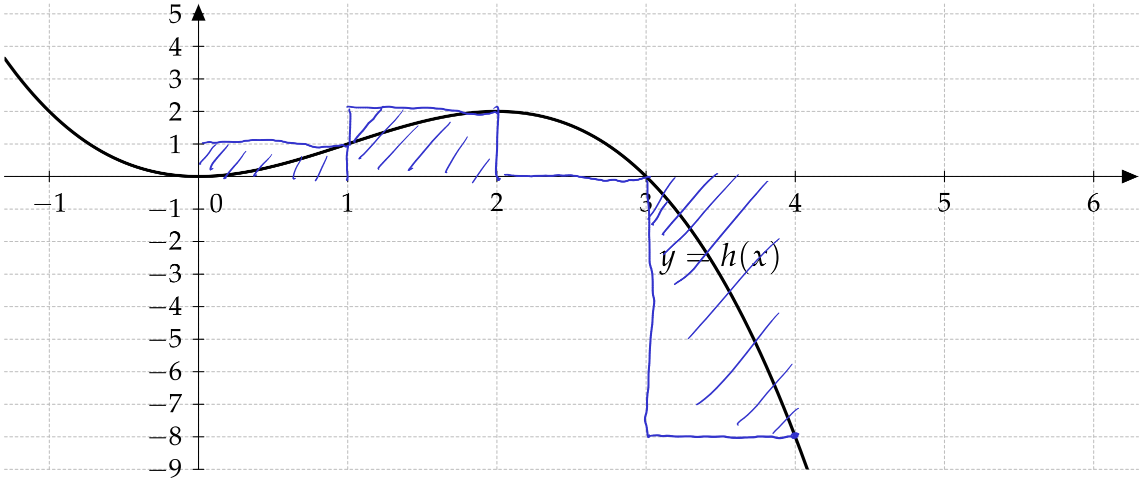
$u = \sin t \qquad t = 0 \to u = 0$

$du = \cos t \, dt \qquad t = \frac{\pi}{2} \to u = 1$

$$= \int_0^1 u^3 \, du = \left.\frac{u^4}{4}\right|_0^1$$

$$= \boxed{\frac{1}{4} \text{ ft}}$$

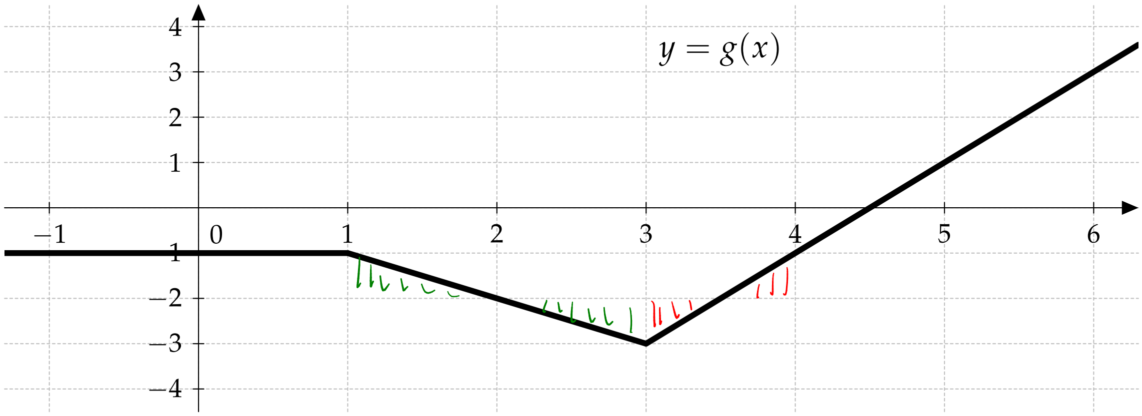
Problem 11. (*5 points*)



$y = h(x)$ is plotted above. Estimate $\int_0^4 h(x)\,dx$ by using a Riemann sum with $n = 4$ subintervals, taking the sampling points to be right endpoints (the Right Hand Rule R_4). Also, illustrate the rectangles on the graph above. (You do not need to evaluate.)

$$\int_0^4 h(x)\,dx \approx 1\cdot(1+2+0-8)$$
$$= \boxed{-5}$$

Problem 12. (*4 points*)

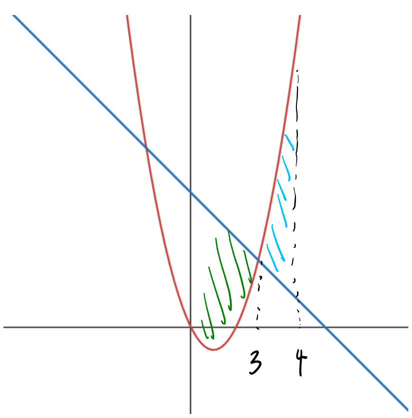


$y = g(x)$ is plotted above. Evaluate the following definite integrals. You do not need to show your work.

i. $\int_2^0 g(x)\,dx = -\int_0^2 g(x)\,dx = -(-2.5) = 2.5$

ii. $\int_1^4 g(x)\,dx = -6$ (pairing up regions) $\int_1^3 = -4$, $\int_3^4 = -2$

Problem 13. (*6 points*) Set up the integral to find the area bounded between $y = x^2 - 2x$ and $y = -x + 6$ between $x = 0$ and $x = 4$.



$x^2 - 2x = -x + 6$

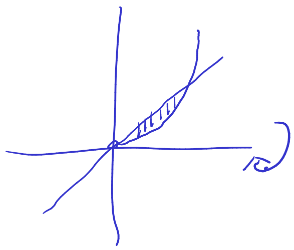
$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3, -2$

$$\boxed{A = \int_0^3 [(-x+6) - (x^2 - 2x)]\,dx + \int_3^4 [(x^2-2x) - (-x+6)]\,dx}$$

Problem 14. (*7 points*) Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ around the x-axis.



washer method

$$V = \pi \int_0^1 \left[(x)^2 - (x^2)^2 \right] dx$$

$$= \pi \int_0^1 (x^2 - x^4)\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$= \boxed{\pi \cdot \frac{2}{15}}$$