Math 220 | Spring 2024

Math 220 Final Exam

Name: Soln

Recitation instructor: ______________________

Recitation time: ______________________

- Exam date/time: May 8, 2024 at 6:20 - 8:10 pm.
- This is a closed-book, closed-notes exam. No calculators or electronic aids are permitted.
- Read each question carefully and show your work unless explicitly told otherwise.
- If you need extra room, use the blank page at the end of this exam. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Grading

1	/15	**8**	/7
2	/10	**9**	/12
3	/3	**10**	/8
4	/5	**11**	/6
5	/6	**12**	/4
6	/5	**13**	/6
7	/6	**14**	/7
Total	/100		

Problem 1. (*15 points*)
(3 points each) Evaluate the following:

A. Use L'Hopital rule to evaluate $\lim_{\theta \to 0} \frac{1 - \cos(\theta^4)}{\sin(\theta^4)}$.

$\frac{1-1}{0}$

$\overset{LH}{=} \lim_{\theta \to 0} \frac{\sin(\theta^4)\,\cancel{4\theta^3}}{\cos(\theta^4)\cdot\cancel{4\theta^3}}$

$= \frac{0}{1} = 0$

B. $\int \left(x^{-1/2} + 3\sec(x)\tan(x) - \frac{1}{\sqrt{1-x^2}} \right) dx =$

$= 2x^{\frac{1}{2}} + 3\sec x - \sin^{-1} x + C$

C. $\frac{d}{dx} \int_3^{x^5} t^2 e^{\cos t}\, dt =$

$= x^{10} e^{\cos(x^5)} \cdot 5x^4$

D. $\frac{d}{dx}\left(\frac{e^x+\log_7(x)}{\arcsin x}\right) =$ $\cancel{\frac{x}{\sqrt{1-x^2}}}$

$$\frac{\arcsin x\left(e^x + \frac{1}{\ln 7}\frac{1}{x}\right) - \left(e^x + \log_7 x\right)\frac{1}{\sqrt{1-x^2}}}{(\sin^{-1} x)^2}$$

E. $\frac{d}{dx}\left(\tan(x^6+6)\cdot 6^x\right) =$ $\sec^2(x^6+6)\cdot 6x^5\cdot 6^x + \tan(x^6+6)(\ln 6)\,6^x$

Problem 2. (*10 points*)

A. (*5 points*) Use implicit differentiation to find $\frac{dy}{dx}$ for $x^2 - 5xy + y^3 = \sec(x)$.

$$2x - 5(y + xy') + 3y^2y' = \sec x \tan x$$

$$y'(-5x + 3y^2) = \sec x \tan x - 2x + 5y$$

$$y' = \frac{\sec x \tan x - 2x + 5y}{-5x + 3y^2}$$

B. (*5 points*) Use logarithmic differentiation to find $\frac{dy}{dx}$ for $y = x^{3\sqrt{x}}$.

$$\ln y = 3\sqrt{x} \ln x$$

$$\frac{y'}{y} = 3 \frac{1}{2\sqrt{x}} \ln x + \frac{3\sqrt{x}}{x}$$

$$y' = 3x^{3\sqrt{x}} \left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} \right)$$

$w'' \; \underset{-1}{\overset{- \quad +}{\longrightarrow}}$

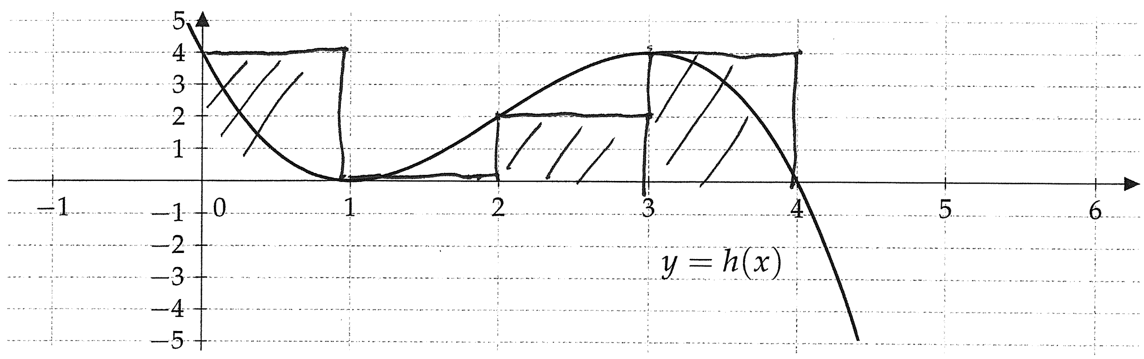
Problem 3. (*3 points*) (1 point each) For the function $w(x)$, one has $w''(x) = (x+1)e^x$. Find the following: (Do not need to show your work).

A. Interval(s) where $w(x)$ is concave up: $(-1, \infty)$

B. Interval(s) where $w(x)$ is concave down: $(-\infty, -1)$

C. x-coordinate(s) where $w(x)$ has an inflection point: $x = -1$

Problem 4. (*5 points*)



$y = h(x)$ is plotted above. Estimate $\int_0^4 h(x)\,dx$ by using a Riemann sum with $n = 4$ subintervals, taking the sampling points to be left endpoints (the Left Hand Rule L_4). Also, illustrate the rectangles on the graph above.

$$\int_0^4 h(x)\,dx \approx 1(4+0+2+4)$$
$$= 10$$

Problem 5. (*6 points*) Use a linearization of $u(x) = 4\sqrt{x}$ at $x = 4$ to approximate $4\sqrt{4.02}$.

$$L(x) = u(4) + u'(4)(x-4)$$

$$u(4) = 4 \cdot 2 = 8$$

$$= 8 + 1(x-4)$$

$$u'(x) = \frac{4}{2\sqrt{x}} = \frac{2}{\sqrt{x}}$$

$$u'(4) = 1$$

$$4\sqrt{4.02} \underset{=}{} u(4.02) \approx L(4.02)$$

$$= 8 + 1(0.02)$$

$$= 8.02$$

Problem 6. (*5 points*) Using the limit definition of the derivative, find $f'(2)$ if $f(x) = x^2 + x$.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 + (2+h) - 6}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 + \cancel{2} + h - \cancel{6}}{h}$$

$$= \lim_{h \to 0} \frac{4h + h^2 + h}{h} = \lim_{h \to 0} h + 5 = 5$$

Check

$$f'(x) = 2x+1$$

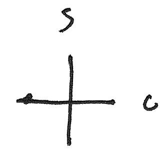
$$f'(2) = 5 \checkmark$$

Problem 7. (*6 points*) Find the absolute minimum and maximum of $w(x) = x + \sin x$ on the interval $[0, 2\pi]$. (Hint: $\pi \approx 3.14$, and $\cos \pi = -1$). You can leave the answer in term of π.

$w'(x) = 1 + \cos x = 0$

$\Rightarrow \cos x = -1$

$x = \pi$ crit point.



$w(0) = 0 + 0 = 0$

$w(\pi) = \pi + 0 = \pi$

$w(2\pi) = 2\pi + 0 = 2\pi$

abs min : 0

abs max : 2π

Problem 8. (*7 points*) Suppose that the length of a rectangle is increasing at a rate of 4 m/s and the width is decreasing at a rate of 5 m/s. How fast is the area changing when the length is 7 m and the width is 10 m ? Is the area increasing or decreasing? (Include units with your answer.)

$A = lw$ $\qquad l = 7\,m \quad l' = 4\,m/s$

$\qquad\qquad\quad w = 10\,m \quad w' = -5\,m/s$

$A' = l'w + lw'$

$= 4 \cdot 10 + 7(-5)$

$= 40 - 35 = 5\ m^2/s$ $\qquad$ area increasing

Problem 9. (*12 points*)
(6 points each) Evaluate the following:

A. $\int \frac{1}{(\ln x)x}\, dx$

$u = \ln x$
$du = \frac{1}{x}\, dx$

$= \int \frac{1}{u}\, du$
$= \ln|u| = \ln|\ln x| + C$

B. $\int_0^1 \frac{x}{1+9x^4}\, dx$

$u = 3x^2$
$du = 6x\, dx$

$x = 0 \to u = 0$
$x = 1 \to u = 3$

$= \frac{1}{6} \int_0^3 \frac{1}{1+u^2}\, du$

$= \frac{1}{6} \tan^{-1}(u) \Big|_0^3$

$= \frac{1}{6} \tan^{-1}(3) - \underbrace{\frac{1}{6} \tan^{-1}(0)}_{=0}$

$= \frac{1}{6} \tan^{-1}(3)$

Problem 10. (*8 points*) An apartment owner can build as many apartment units as she wants. Suppose that 400 apartment units will be rent if its rent is \$300 per month. Suppose that each time rent is decreased by \$10, twenty additional units will be rent. What should the manager charge for rent to maximize income? (Make sure to justify why your answer corresponds to the absolute maximum.) (Include units with your answer.)

$a(r) = 400 - 2(r-300)$

$a(300) = 400$ rent $\downarrow 10$, $a(r) \uparrow 20$

$i(r) = a(r) \cdot r$

$= 400r - 2r(r-300)$

$= -2r^2 + 1000r$

$i'(r) = -4r + 1000 = 0$ Claim

$\Rightarrow r = 250\,\$ \leftarrow$ this maximises income

Justif: $\underline{\text{2nd D.T.}}$

$i'' = -4 < 0 \therefore$ crit point is abs max.

$\underline{\text{1st D.T.}}$

I' $+ \mid -$ 250

Problem 11. (*6 points*) An insect population is increasing at a rate of te^{t^2} insects/day on day t. Find the size of the insect population after 4 days assuming that there are 10 insects at time $t = 0$ days. (Include units with your answer.) You can leave your answer in terms of power of e.

$$p'(t) = te^{t^2}$$

$$p(0) = 10$$

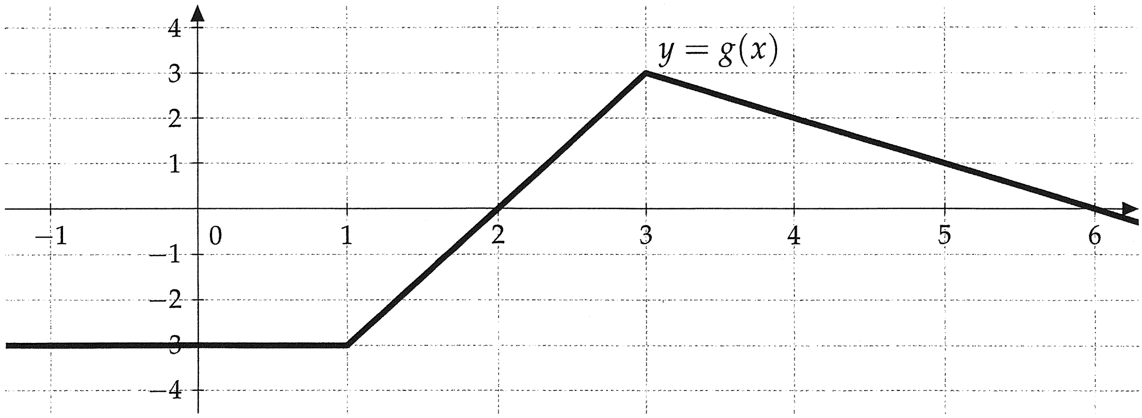
u-sub w/ $\cancel{t} u = t^2$

$$\cancel{p(4) =} \quad p(4) - p(0) = \int_0^4 p'(t)\,dt = \int_0^4 te^{t^2}\,dt = \frac{1}{2} e^{t^2} \Big|_0^4$$

$$= \frac{1}{2}(e^{16} - 1)$$

$$\therefore \quad p(4) = \frac{1}{2}(e^{16} - 1) + 10 \text{ insects}$$

Problem 12. (*4 points*)

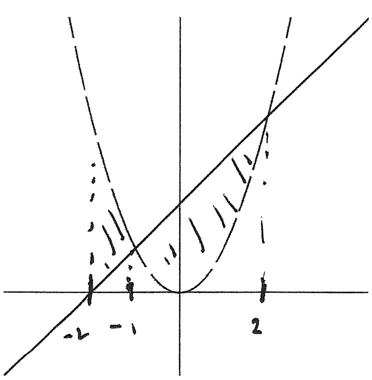


$y = g(x)$ is plotted above. Evaluate the following definite integrals. (Do not need to show your work).

i. $\int_{1}^{-1} g(x)\,dx =$ 6

ii. $\int_{1}^{5} g(x)\,dx =$ $\frac{1}{2}(2)(3+1) = 4$

Problem 13. (*6 points*)
Set up the integral to find the area bounded between $y = x^2$ and $y = x + 2$ between $x = -2$ and $x = 2$. [Do <u>not</u> evaluate the integral you derive.]



$x^2 = x+2$

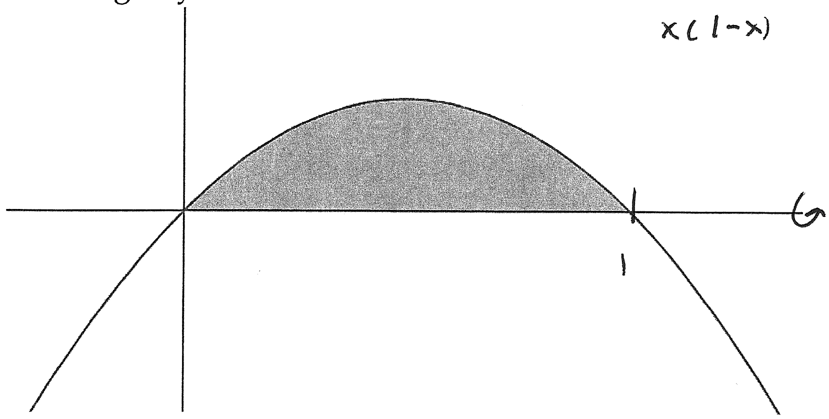
$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$A = \int_{-2}^{-1} x^2 - (x+2)\,dx + \int_{-1}^{2} x+2 - x^2\,dx$

Problem 14. (*7 points*) Find the volume of the solid obtained by rotating the shaded region (see figure below) bounded by $y = x - x^2$ and $y = 0$ around the x-axis. Evaluate the integral you derive.

$x(1-x)$



Disk method

$$V = \pi \int_0^1 (x - x^2)^2 \, dx$$

$$= \pi \int_0^1 x^2 - 2x^3 + x^4 \, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right)$$

$$\frac{2-3}{6} = \frac{-1}{6}$$

$$= \boxed{\frac{1}{30} \pi}$$

$$\frac{-1}{6} + \frac{1}{5} = \frac{-5+6}{30} = \frac{1}{30}$$