Math 221 Calculus 2
Professor John Maginnis

Midterm Exam 1
September 24, 2013

Your name: Solutions

Rec. Instr.: ______

Rec. Time: ______

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/10	/10	/10	/10
Problem	5	6		Total
Points	/10	/10		/60

1. Evaluate the limit.

$$\lim_{x\to\infty}\left(\frac{x-1}{x}\right)^x$$

Let $y = \left(\frac{x-1}{x}\right)^x = \left(1-\frac{1}{x}\right)^x$.

Then $\ln(y) = \ln\left(\left(1-\frac{1}{x}\right)^x\right) = x\ln\left(1-\frac{1}{x}\right)$.

$$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \frac{\ln(1-0)}{0} = \frac{0}{0},$$

so we may use <u>L'Hôpital's Rule</u>:

$$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty}\frac{\left(\frac{1/x^2}{1-\frac{1}{x}}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty}\frac{-1}{1-\frac{1}{x}} = \frac{-1}{1-0}$$

Finally, since $\lim_{x\to\infty}\ln(y) = -1$,

we have $\lim_{x\to\infty}\left(\frac{x-1}{x}\right)^x = \lim_{x\to\infty}(y) = e^{\lim_{x\to\infty}\ln(y)}$

$$= e^{-1} = \boxed{\frac{1}{e}}$$

2. Evaluate the definite integral.

$$\int_0^{\frac{\pi}{3}} \sec^4(\theta)\tan^3(\theta)\; d\theta$$

<u>Method 1</u>: Substitute $u = \tan\theta$, $du = \sec^2\theta\, d\theta$, and use Pythagorean's Theorem $\sec^2\theta = 1 + \tan^2\theta$.

$$\int_0^{\sqrt{3}} (1+u^2)\,u^3\,du = \int_0^{\sqrt{3}} (u^3+u^5)\,du = \left[\frac{1}{4}u^4 + \frac{1}{6}u^6\right]_0^{\sqrt{3}}$$

$$= \frac{9}{4} + \frac{27}{6} - (0+0) = \frac{9}{4} + \frac{9}{2} = \boxed{\frac{27}{4}}$$

Note $\frac{1}{4}u^4 + \frac{1}{6}u^6 = \frac{1}{4}\tan^4(\theta) + \frac{1}{6}\tan^6(\theta)$.

<u>Method 2</u>: Substitute $v = \sec\theta$, $dv = \sec\theta\tan\theta\, d\theta$, and $\tan^2\theta = \sec^2\theta - 1$. Then we obtain

$$\int_1^2 v^3(v^2-1)\,dv = \int_1^2 (v^5 - v^3)\,dv = \left[\frac{1}{6}v^6 - \frac{1}{4}v^4\right]_1^2 =$$

$$\left(\frac{64}{6} - \frac{16}{4}\right) - \left(\frac{1}{6} - \frac{1}{4}\right) = \frac{63}{6} - \frac{15}{4} = \frac{21}{2} - \frac{15}{4} = \boxed{\frac{27}{4}}$$

Note $\frac{1}{6}v^6 - \frac{1}{4}v^4 = \frac{1}{6}\sec^6(\theta) - \frac{1}{4}\sec^4(\theta)$.

3. Evaluate the integral.

$$\int \frac{7x-6}{x^3-2x^2}\,dx$$

Factor $x^3-2x^2 = x^2(x-2)$.

Partial Fractions $\dfrac{7x-6}{x^3-2x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$

$7x-6 = Ax(x-2) + B(x-2) + Cx^2$

If $x=0$, $-6=-2B$, so $B=3$.

If $x=2$, $8=4C$, so $C=2$.

Comparing coefficients of x^2 yields $0=A+C$, so $A=-2$.

$$\int \frac{7x-6}{x^3-2x^2}\,dx = \int \frac{-2}{x} + \frac{3}{x^2} + \frac{2}{x-2}\,dx$$

$$= \boxed{-2\ln|x| - \frac{3}{x} + 2\ln|x-2| + C}$$

$$= 2\ln\left|\frac{x-2}{x}\right| - \frac{3}{x} + C$$

4. Evaluate the improper integral.

$$\int_0^\infty x^2 e^{-x}\,dx = \lim_{R\to\infty} \int_0^R x^2 e^{-x}\,dx$$

Parts $u = x^2, \; dv = e^{-x}dx$

$du = 2x\,dx, \; v = -e^{-x}$

$\int u\,dv = uv - \int v\,du$

$$\int x^2 e^{-x}dx = -x^2 e^{-x} + \int 2x e^{-x}dx$$

Parts again $u = 2x, \; dv = e^{-x}dx$

$du = 2\,dx, \; v = -e^{-x}$

$$\int x^2 e^{-x}dx = -x^2 e^{-x} - 2x e^{-x} + \int 2e^{-x}dx$$

$$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

$$\int_0^R x^2 e^{-x}dx = \left(-R^2 e^{-R} - 2Re^{-R} - 2e^{-R}\right) - (-2)$$

$$\lim_{R\to\infty}\left(\frac{-R^2-2R-2}{e^R}\right) = \frac{\infty}{\infty} = \lim_{R\to\infty}\left(\frac{-2R-2}{e^R}\right) = \frac{\infty}{\infty} =$$

$$\lim_{R\to\infty}\left(\frac{-2}{e^R}\right) = \frac{-2}{\infty} = 0.$$ Thus $\int_0^\infty x^2 e^{-x}dx = \boxed{2}$

5. Evaluate the integral.

$$\int 6x^2 \tan^{-1}(x)\, dx$$

<u>Parts</u> $u = \tan^{-1}(x)$, $dv = 6x^2 dx$

$du = \frac{1}{1+x^2} dx$, $v = 2x^3$

$$\int u\,dv = uv - \int v\,du$$

$$\int 6x^2 \tan^{-1}(x)\,dx = 2x^3 \tan^{-1}(x) - \int \frac{2x^3}{1+x^2}\,dx$$

<u>long division</u> $\frac{2x^3}{1+x^2} = 2x - \frac{2x}{1+x^2}$

$$\int 6x^2 \tan^{-1}(x)\,dx = 2x^3 \tan^{-1}(x) - \int 2x\,dx + \int \frac{2x}{1+x^2}\,dx$$

$$= \boxed{2x^3 \tan^{-1}(x) - x^2 + \ln(1+x^2) + C}$$

Note here we used:

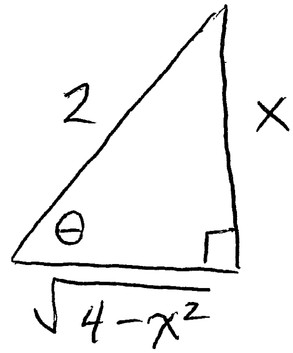
$w = 1+x^2$

$dw = 2x\,dx$

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{dw}{w} = \ln|w| + C$$

6. Evaluate the integral.

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$



$$\sin\theta = \frac{x}{2}$$

$$\cos\theta = \frac{\sqrt{4-x^2}}{2}$$

Substitute $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = \int \frac{2\cos\theta}{4\sin^2\theta}\cdot 2\cos\theta\, d\theta =$$

$$\int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2(\theta)\,d\theta$$

Pythagorean Theorem $\cot^2(\theta) = \csc^2(\theta) - 1$

$$\int \cot^2(\theta)\,d\theta = \int \csc^2\theta\,d\theta - \int d\theta$$

$$= -\cot(\theta) - \theta + C$$

$$= \boxed{-\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C}$$