Math 221 Calculus 2
Professor John Maginnis

Midterm Exam 2
October 22, 2013

Your name: Solutións

Rec. Instr.: __________ Rec. Time: __________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/10	/10	/10	/10
Problem	5	6		Total
Points	/10	/10		/60

1. Find the arc length of the curve $y = 2x\sqrt{x} = 2x^{\frac{3}{2}}$ for $0 \le x \le 7$.

$$\frac{dy}{dx} = \left(\frac{3}{2}\right) 2x^{\frac{1}{2}} = 3x^{1/2} = 3\sqrt{x}$$

$$\left(\frac{dy}{dx}\right)^2 = 9x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 9x$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+9x}\, dx$$

$$s = \int_0^7 \sqrt{1+9x}\, dx = \int_1^{64} \frac{1}{9}\sqrt{u}\, du = \left[\frac{1}{9}\,\frac{u^{3/2}}{3/2}\right]$$

Substitute $u = 1+9x$

$du = 9\, dx$

$\frac{1}{9} du = dx$

$$= \left[\frac{2}{27}(1+9x)^{3/2}\right]_0^7$$

$$s = \frac{2}{27}(64)^{3/2} - \frac{2}{27}(1)^{3/2} = \frac{2}{27}(512) - \frac{2}{27}$$

$$s = \frac{1024-2}{27} = \boxed{\frac{1022}{27}}$$

2. Find the centroid of the region under the curve $y = \sin(x)$ for $0 \leq x \leq \pi$.

$$M = A = \int_0^{\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi} = 1 - (-1) = \boxed{2}$$

$$M_y = \int_0^{\pi} x \sin(x)\,dx = \Big[-x\cos(x)\Big]_0^{\pi} - \int_0^{\pi} -\cos(x)\,dx$$

Parts $\quad u = x \quad dv = \sin(x)\,dx$

$\quad du = dx \quad v = -\cos(x)$

$$M_y = \Big[-x\cos(x) + \sin(x)\Big]_0^{\pi} = -\pi(-1) + 0 - (-0+0)$$

$$= \boxed{\pi}$$

$$M_x = \frac{1}{2}\int_0^{\pi} (\sin(x))^2\,dx = \frac{1}{2}\int_0^{\pi} \frac{1-\cos(2x)}{2}\,dx$$

$$M_x = \Big[\frac{1}{4}x - \frac{\sin(2x)}{8}\Big]_0^{\pi} = \left(\frac{\pi}{4} - 0\right) - (0-0) = \boxed{\frac{\pi}{4}}$$

Thus $\bar{x} = \dfrac{M_y}{M} = \boxed{\dfrac{\pi}{2}}$ (Note this can be deduced from the symmetry of the sine curve.)

$$\bar{y} = \frac{M_x}{M} = \frac{\pi/4}{2} = \boxed{\frac{\pi}{8}} \quad \text{Centroid} = \left(\frac{\pi}{2}, \frac{\pi}{8}\right).$$

3. (a) Find the limit of the sequence $a_n = n^2(1-\cos(\frac{1}{n})) = \frac{1-\cos(\frac{1}{n})}{\frac{1}{n^2}}$.

$$\lim_{n\to\infty}(a_n) = \lim_{x\to\infty} \frac{1-\cos\left(\frac{1}{x}\right)}{\frac{1}{x^2}} = \frac{0}{0} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)\cdot\frac{-1}{x^2}}{\frac{-2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{2}{x}} = \frac{0}{0} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\frac{-1}{x^2}}{\frac{-2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{2}\cos\left(\frac{1}{x}\right) = \boxed{\frac{1}{2}}$$ (using L'Hôpital's Rule twice)

(b) Determine whether the series converges. Explain.

$\sum_{n=1}^{\infty} n^2(1-\cos(\frac{1}{n}))$ This series diverges since $\lim_{n\to\infty}(a_n)$ is not zero.

(c) Find the sum of the series.

$$\sum_{n=0}^{\infty} \frac{2^{n+2}-3^n}{5^{n+1}} = \sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{2}{5}\right)^n - \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{3}{5}\right)^n$$

$$= \frac{4/5}{1-2/5} - \frac{1/5}{1-3/5} = \frac{4}{3} - \frac{1}{2} = \boxed{\frac{5}{6}}$$

using the formula for the sum of a geometric series

$$\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r} \text{ if } -1<r<1.$$

4. Approximate $\sqrt{1.04}$ as follows.

(a) Find the second Taylor polynomial $T_2(x)$ for $f(x) = \sqrt{x}$ at $a = 1$.

$f(x) = x^{1/2}$ $\qquad f(1) = \sqrt{1} = 1$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = \frac{-1}{4} x^{-3/2}$ $\qquad f''(1) = \frac{-1}{4}$

$$T_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1/4}{2}(x-1)^2$$

$$\boxed{T_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2}$$

(b) Evaluate $T_2(1.04)$.

$$T_2(1.04) = 1 + \frac{1}{2}(.04) - \frac{1}{8}(.04)^2$$

$$= 1 + .02 - \frac{.0016}{8} = 1.02 - .0002 = \boxed{1.0198}$$

(c) Estimate the error using the formula $K\frac{|x-a|^{n+1}}{(n+1)!}$.

$f'''(x) = \frac{3}{8} x^{-5/2}$ is positive, and is decreasing since

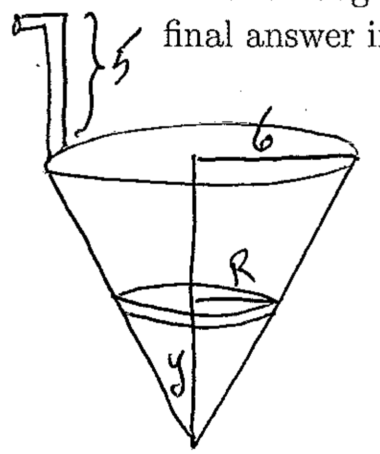
$f^{(4)}(x) = \frac{-15}{16} x^{-7/2}$ is negative. Thus $K = f'''(1) = \frac{3}{8}$

satisfies $|f'''(c)| \le \frac{3}{8}$ for $1 \le c \le 1.04$.

$$|f(1.04) - T_2(1.04)| \le \frac{3}{8} \cdot \frac{(.04)^3}{6} = \frac{.000064}{16}$$

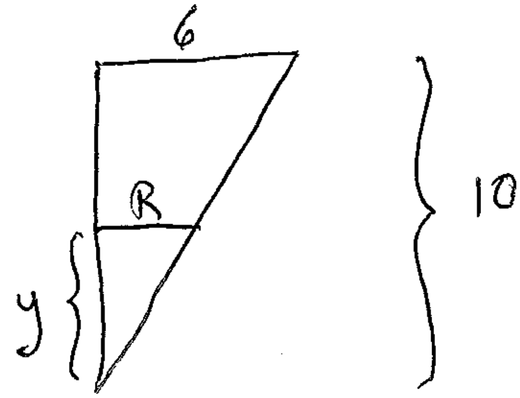
$$= \boxed{.000004}$$

5. A tank full of water has the shape of a cone of height 10 meters, and the radius at the top of the tank is 6 meters. Find the work done in pumping out the water through a spout which is 5 meters above the top of the tank. Write your final answer in terms of g and ρ. Do <u>not</u> plug in $g = 9.8\frac{m}{sec^2}$ and $\rho = 1000\frac{kg}{m^3}$.



similar triangles

$$\frac{R}{6} = \frac{y}{10}$$



$$dV = \pi R^2 dy \qquad R = \frac{6}{10}y = \frac{3}{5}y$$

the distance this cross section needs to be pumped is $15 - y$, if y = height above bottom of tank.

$$W = \int_0^{10} \rho \cdot g \cdot \pi \cdot (15-y) \cdot \left(\frac{3}{5}y\right)^2 dy$$

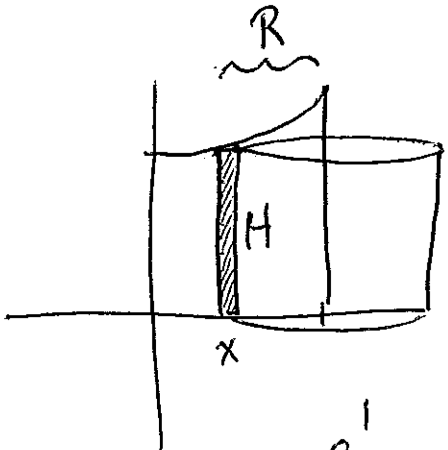
$$= \frac{9\pi}{25}\rho g \int_0^{10} (15y^2 - y^3)\,dy$$

$$= \frac{9\pi}{25}\rho g \left[5y^3 - \frac{1}{4}y^4\right]_0^{10} = \frac{9\pi}{25}\rho g \left[5000 - \frac{10000}{4}\right]$$

$$= \frac{9\pi}{25}\rho g (5000 - 2500) = \frac{9\pi}{25}\rho g (2500)$$

$$= \boxed{900\pi\rho g \text{ Joules}}$$

6. Find the volume of revolution formed by revolving the region under the curve $y = \frac{4}{4-x^2}$ for $0 \le x \le 1$ around the line $x = 1$.



Cylindrical Shells

$$dV = 2\pi R H \, dx$$

$$H = y = \frac{4}{4-x^2}$$

$$R = 1 - x$$

$$V = 2\pi \int_0^1 (1-x)\frac{4}{4-x^2}\,dx = 2\pi \int_0^1 \frac{4-4x}{4-x^2}\,dx$$

Partial Fractions

$$\frac{4-4x}{4-x^2} = \frac{A}{2+x} + \frac{B}{2-x}$$

$$4-x^2 = (2+x)(2-x)$$

$$4-4x = A(2-x) + B(2+x)$$

If $x=2$, $-4 = 4B$, $B = -1$

If $x=-2$, $12 = 4A$, $A = 3$

$$V = 2\pi \int_0^1 \frac{4-4x}{4-x^2}\,dx = 2\pi \int_0^1 \frac{3}{2+x} + \frac{-1}{2-x}\,dx$$

$$V = 2\pi \Big[3\ln|2+x| + \ln|2-x| \Big]_0^1$$

$$= 2\pi\big(3\ln 3 + \ln(1) - (3\ln 2 + \ln 2)\big)$$

$$= \boxed{2\pi(3\ln 3 - 4\ln 2)}$$