Math 221 Calculus 2
Professor John Maginnis

Final Exam
December 18, 2013

Your name: Solutions

Rec. Instr.: ____________________

Rec. Time: ____________________

Instructions:

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators.

For each test of convergence that you use, either give the name of the test, or briefly describe what the test says.

This exam is worth 120 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4	5
Points	/10	/10	/10	/10	/10
Problem	6	7	8	9	10
Points	/10	/10	/10	/10	/10
Problem	11	12			Total
Points	/10	/10			/120

1. Determine whether the series converges or diverges. Explain.

$$\sum_{n=0}^{\infty} \frac{n!}{e^{n^2}}$$

Use the Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{e^{n^2}}{e^{(n+1)^2}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \cdot \frac{e^{n^2}}{e^{n^2+2n+1}} \right| = \lim_{n\to\infty} \left| (n+1) \cdot \frac{1}{e^{2n+1}} \right|$$

Use L'Hôpital's Rule for

$$\lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = \frac{\infty}{\infty} = \lim_{n\to\infty} \frac{1}{2e^{2n+1}} = \frac{1}{\infty} = 0.$$

Since $\rho = 0 < 1$, the series converges.

2. Evaluate the indefinite integral.

$$\int \frac{3x-2}{x^3+x}\,dx$$

Use Partial Fractions

Note $x^3+x = x(x^2+1)$, so that

$$\frac{3x-2}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$3x-2 = A(x^2+1) + Bx^2 + Cx$$

If $x=0$, $-2=A$ so $A=-2$.

The coefficient of x^2 yields $0=A+B$, so $B=2$.

The coefficient of x yields $3=C$.

$$\int \frac{3x-2}{x^3+x}\,dx = \int \frac{-2}{x} + \frac{2x+3}{x^2+1}\,dx$$

$$= \int \frac{-2}{x} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\,dx$$

$$= \boxed{-2\ln|x| + \ln|x^2+1| + 3\tan^{-1}(x) + C}$$

here we use: $u=x^2+1$, $du=2x\,dx$, $\int \frac{du}{u} = \ln|u| + C$

3. Determine whether the series converges or diverges. Explain.

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$$

Note $\frac{n}{\sqrt{n^4+1}} \approx \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}$

Use the limit comparison test:

$$L = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{\sqrt{n^4+1}}} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2} = \lim_{n\to\infty} \sqrt{\frac{n^4+1}{n^4}}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n^4}} = \sqrt{1+0} = \boxed{1}.$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by the p-series test with $p=1$), the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$ also diverges.

Remark: $\frac{n}{\sqrt{n^4+1}} < \frac{1}{n}$ which means that we cannot use the comparison test.

4. Given the following equation in polar coordinates, convert to an equation in rectangular coordinates.

$$r = \frac{1}{1+\cos(\theta)}$$

$r(1+\cos(\theta)) = 1$

$r + r\cos\theta = 1$ Note $x = r\cos\theta$.

$r + x = 1$

$r = 1 - x$ Square both sides.

$r^2 = (1-x)^2 = 1 - 2x + x^2$

$x^2 + y^2 = 1 - 2x + x^2$ Note $r^2 = x^2 + y^2$.

$\boxed{y^2 = 1 - 2x}$ or $x = \frac{1}{2} - \frac{1}{2}y^2$

(a parabola facing to the left)

5. Find the Taylor series at $c = 2$ for the function. You need to find a formula for the general term.

$$f(x) = \frac{1}{x} = x^{-1}$$
$$f'(x) = -x^{-2}$$
$$f''(x) = 2x^{-3}$$
$$f'''(x) = -6x^{-4}$$
$$f^{(4)}(x) = 24x^{-5}$$

Find a pattern in the derivatives:

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

Plug in $c=2$

$$f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}$$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^{n+1}\, n!}(x-2)^n$$

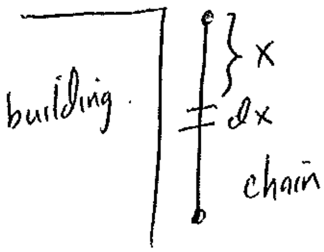
$$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n}$$

<u>Alternative Method</u>: $f(x) = \frac{1}{x} = \frac{1}{2+(x-2)} =$

$$\frac{1}{2}\left(\frac{1}{1+\frac{x-2}{2}}\right) = \frac{1}{2}\left(\frac{1}{1-\left(-\frac{(x-2)}{2}\right)}\right) =$$

$$\frac{1}{2}\underbrace{\sum_{n=0}^{\infty}\left(-\frac{(x-2)}{2}\right)^n}_{\text{geometric series with } r = -\left(\frac{x-2}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{2^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{2^{n+1}}}$$

6. A chain 20 meters long is hanging from the roof of a building. The chain has a density of 5 kilograms per meter. Find the work required to lift the chain to the top of the building. Recall that $g = 9.8 \frac{m}{sec^2}$ is the acceleration of gravity.



For a small piece of chain, let
x = distance to the top of the building.

For this small piece of chain, use

① mass = density × length, $dm = 5\,dx$

② Force = mass × acceleration, $dF = (9.8)(5\,dx)$
$dF = 49\,dx$

③ Work = force × distance, $dW = (x)(49\,dx) = 49x\,dx$

Then the total amount of work is the integral

$$W = \int_0^{20} 49x\,dx = \left[\frac{49}{2}x^2\right]_0^{20} = \frac{49}{2}(400 - 0)$$

$$W = 49(200) = \boxed{9800 \text{ Joules}}$$

7. Evaluate the indefinite integral.

$$\int \sin^3(x)\cos^3(x)\,dx$$

Use the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$.

Also Pythagorean's Thm yields $\cos^2(x) = 1 - \sin^2(x)$.

Then $\int \sin^3(x)\cos^3(x)\,dx = \int \sin^3(x)\cos^2(x)\cos(x)\,dx$

$= \int \sin^3(x)\left(1-\sin^2(x)\right)\cos(x)\,dx = \int u^3(1-u^2)\,du =$

$\int u^3 - u^5\,du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$

$= \boxed{\frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C.}$

A second method uses $V = \cos(x)$ and $dV = -\sin(x)\,dx$

which leads to $\int -V^3 + V^5\,dV = \boxed{\frac{-1}{4}\cos^4(x) + \frac{1}{6}\cos^6(x) + C.}$

A third method uses $\sin(2x) = 2\sin(x)\cos(x)$, so

$\int \sin^3(x)\cos^3(x)\,dx = \frac{1}{8}\int \sin^3(2x)\,dx$. The substitution

$w = \cos(2x)$ with $dw = -2\sin(2x)\,dx$ leads to

$-\frac{1}{16}\int(1-w^2)\,dw = \boxed{\frac{-1}{16}\cos(2x) + \frac{1}{48}\cos^3(2x) + C.}$

8. Find the arc length of the curve described by the parametrization.

$x = 3t^2+5$, $y = 2t^3$, for $0 \le t \le 1$.

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 6t^2$$

$$\left(\frac{dx}{dt}\right)^2 = 36t^2 \qquad \left(\frac{dy}{dt}\right)^2 = 36t^4$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$ds = \sqrt{36t^2+36t^4}\, dt = \sqrt{36t^2(1+t^2)}\, dt$$

$$ds = 6t\sqrt{1+t^2}\, dt \qquad (\text{note } t \geq 0)$$

$$s = \int_0^1 6t\sqrt{1+t^2}\, dt \qquad \text{Substitute } u = 1+t^2, \; du = 2t\, dt$$

$$s = \int_1^2 3\sqrt{u}\, du = \left[\frac{3u^{3/2}}{3/2}\right]_1^2 = \left[2u^{3/2}\right]_1^2$$

$$= 2(2\sqrt{2}-1) = \boxed{4\sqrt{2}-2}$$

9. Find an equation of the tangent line to the curve given by the parametrization

$$x = t\cos(\frac{\pi}{t}), \quad y = \frac{t^2}{t^2+1} \quad \text{when } t = 1.$$

Note that when $t=1$, $x = (1)\cos(\pi) = -1$ and $y = \frac{1}{1+1} = \frac{1}{2}$. So the point of tangency is $(-1, \frac{1}{2})$.

$$\frac{dx}{dt} = (1)\cdot\cos\left(\frac{\pi}{t}\right) + t\cdot\left(-\sin\left(\frac{\pi}{t}\right)\right)\left(\frac{-\pi}{t^2}\right)$$

$$= \boxed{\cos\left(\frac{\pi}{t}\right) + \frac{\pi}{t}\sin\left(\frac{\pi}{t}\right)}$$

So when $t=1$, $\frac{dx}{dt} = \cos(\pi) + \pi\sin(\pi) = -1$.

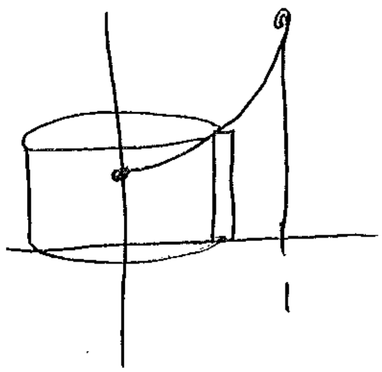
$$\frac{dy}{dt} = \frac{(t^2+1)(2t) - (t^2)(2t)}{(t^2+1)^2} = \boxed{\frac{2t}{(t^2+1)^2}}$$

So when $t=1$, $\frac{dy}{dt} = \frac{2}{(2)^2} = \frac{1}{2}$.

Then the slope $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1/2}{-1} = \frac{-1}{2}$.

$\boxed{y - \frac{1}{2} = \frac{-1}{2}(x+1)}$ or $\boxed{y = \frac{-1}{2}x}$ is the tangent line.

10. Find the volume of revolution if the region below the curve $y = e^{3x}$ between $x = 0$ and $x = 1$ is revolved around the y-axis.



Cylindrical shells

$$dV = 2\pi R H \, dx$$

$$R = x$$

$$H = y = e^{3x}$$

$$V = \int_0^1 2\pi x e^{3x} \, dx = \left[\frac{2\pi}{3} x e^{3x} \right]_0^1 - \int_0^1 \frac{2\pi}{3} e^{3x} \, dx$$

Parts $u = 2\pi x$ $\quad dv = e^{3x} dx$

$du = 2\pi \, dx$ $\quad v = \frac{1}{3} e^{3x}$

$$V = \left[\frac{2\pi}{3} x e^{3x} - \frac{2\pi}{9} e^{3x} \right]_0^1 =$$

$$\frac{2\pi}{3} e^3 - \frac{2\pi}{9} e^3 - \left(0 - \frac{2\pi}{9} \right) =$$

$$\frac{6\pi}{9} e^3 - \frac{2\pi}{9} e^3 + \frac{2\pi}{9} = \boxed{\frac{4\pi}{9} e^3 + \frac{2\pi}{9}}$$

$$= \frac{2\pi (2e^3 + 1)}{9}$$

11. Determine whether the series converges or diverges. Explain.

$$\sum_{n=1}^{\infty} \frac{n}{2n-1}$$

This series Diverges by the Divergence Theorem (or the n^{th} term test).

since $\lim_{n\to\infty} \frac{n}{2n-1} = \boxed{\frac{1}{2}}$ which is not equal to zero.

12. Find the total area inside the four loops of

$r^2 = \sin(4\theta)$.

θ	$r^2 = \sin(4\theta)$	r
0	0	0
$\pi/8$	1	± 1
$\pi/4$	0	0
$3\pi/8$	-1	Does not exist
$\pi/2$	0	0

Note that $r = 0$ when $\sin(4\theta) = 0$ so that

$4\theta = 0, \pi, 2\pi, 3\pi$, etc.

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, etc.

Also $\sin(4\theta) > 0$ if $0 < \theta < \frac{\pi}{4}$, $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$, etc.

One of the four loops occurs in the first quadrant with $0 \le \theta \le \frac{\pi}{4}$. The area of this one loop is:

$$\int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/4} \frac{1}{2}\sin(4\theta)\, d\theta = \left[\frac{-\cos(4\theta)}{8}\right]_0^{\pi/4}$$

$$= \frac{-\cos(\pi)}{8} - \left(\frac{-\cos(0)}{8}\right) = \frac{1}{8} + \frac{1}{8} = \boxed{\frac{1}{4}}.$$

By symmetry the total area of four loops is

$$4 \cdot \left(\frac{1}{4}\right) = \boxed{1}.$$