Calculus 2, Exam 2, March 5 2013

Name
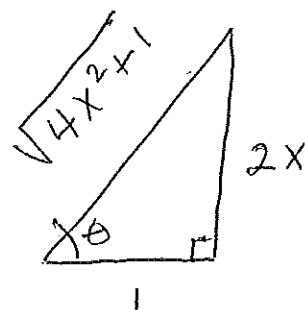
Recitation Instructor and Hour

In order to receive full credit (or any credit it all), answers must be justified, so that they won't be confused with guesse. Drawings (correctly labelled) that bear on a given problem are good sources of partial credit.

(15) Problem 1. Evaluate the integral $I = \int cosh^5 x \, dx$. (A possible hint: How would you find $\int cos^5 x \, dx$ without using a "reduction formula" ?)

$$I = \int cosh^5 x \, dx = \int cosh^4 x \; \underline{cosh \, x \, dx}$$

$$= \int (sinh^2 x + 1)^2 \; cosh \, x \, dx \qquad du \; for$$
$$u = sinh \, x$$

$$= \int (u^2+1)^2 \, du = \int (u^4 + 2u^2 + 1) \, du$$

$$= \boxed{\frac{1}{5} sinh^5 x + \frac{2}{3} sinh^3 x + sinh \, x + C}$$

(20) **Problem 2.** Evaluate the integral

$$I = \int \frac{\sqrt{4x^2+1}}{x}\,dx$$

From the triangle

$$2x = \tan\theta$$

$$x = \frac{1}{2}\tan\theta$$

$$dx = \frac{1}{2}\sec^2\theta\,d\theta$$

and $\sqrt{4x^2+1} = \sec\theta$

So —

$$I = \int \frac{\sec\theta}{\frac{1}{2}\tan\theta}\cdot\frac{1}{2}\sec^2\theta\,d\theta$$

$$= \int \frac{\sec^3\theta\,d\theta}{\tan\theta}$$

(Now it gets tricky.)

$$= \int \frac{\sec^2\theta\,\sec\theta}{\tan\theta}\,d\theta = \int \frac{(1+\tan^2\theta)\sec\theta}{\tan\theta}\,d\theta$$

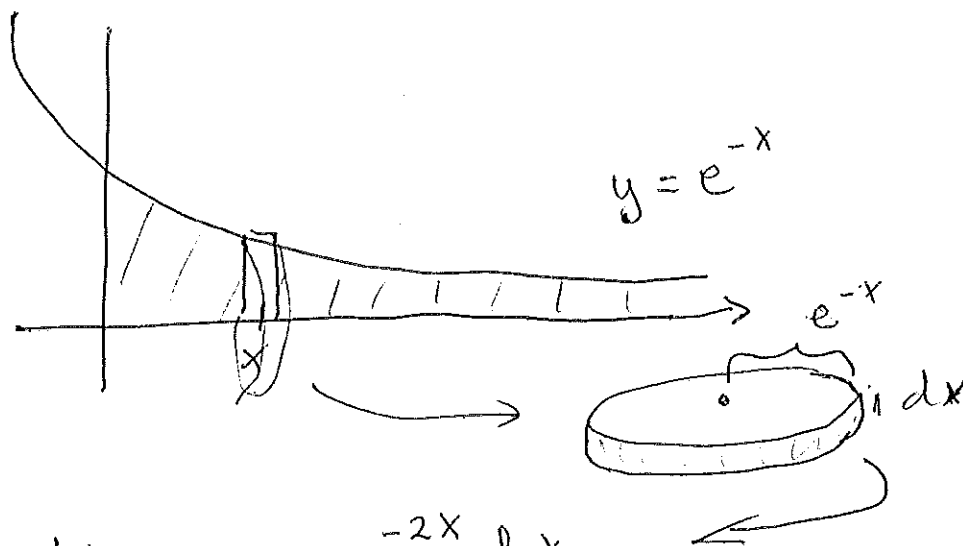$$= \int \frac{\sec\theta}{\tan\theta}\,d\theta + \int \tan\theta\sec\theta\,d\theta$$

$$= \int \csc\theta\,d\theta + \sec\theta + C$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C$$

$$= \boxed{\ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + \sqrt{4x^2+1} + C}$$

(From the triangle)

(15) Problem 3. Let $\mathcal{R}$ be the region of the plane bounded by the graph of $y = e^{-x}$ and the positive coordinate axes. (So, $\mathcal{R}$ extends infinitely far to the right.) Find the volume of the "solid of revolution" obtained by revolving $\mathcal{R}$ about the x-axis.

$$dV = \pi e^{-2x}\, dx$$

$$V = \pi \int_0^\infty e^{-2x}\, dx = -\frac{1}{2}\pi e^{-2x}\Big|_0^\infty$$

$$= \frac{\pi}{2} e^{-2x}\Big|_\infty^0 = \frac{\pi}{2}\frac{1}{e^{2x}}\Big|_\infty^0$$

$$= \frac{\pi}{2}\left(\frac{1}{e^0} - \lim_{x\to\infty}\frac{1}{e^{2x}}\right)$$

$$= \frac{\pi}{2}\left(1 - 0\right) = \boxed{\frac{\pi}{2}}$$

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(20) **Problem 4.** Evaluate the integral

$$I = \int \frac{2x \, dx}{(x-1)(x^2+1)}.$$

(For the solution of this problem, it may be helpful to know that $\int \frac{du}{u^2+1} = \tan^{-1}u + C.$)

1st $\quad \dfrac{2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$\Rightarrow \quad 2x = A(x^2+1) + (Bx+C)(x-1)$

When $x=1$ get $\quad 2 = 2A$, so $\boxed{A=1}$

Coefficient of x^2: $\quad 0 = A+B$, so $\boxed{B=-1}$

Coefficient of x: $\quad 2 = -B+C$, so $\boxed{C=3}$

2nd $\quad I = \int \dfrac{dx}{x-1} - \int \dfrac{x \, dx}{x^2+1} + \int \dfrac{3 \, dx}{x^2+1}$

$\left(u = x^2+1 , \quad dx = \frac{1}{2} du \right)$

$I = \ln|x-1| - \frac{1}{2}\int \dfrac{du}{u} + 3\tan^{-1}x + C$

$\boxed{I = \ln|x-1| - \frac{1}{2}\ln|x^2+1| + 3\tan^{-1}x + C}$

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(15) **Problem 5.** Let L be the line segment given by the graph of $y = 1 - x$ from $x = 0$ to $x = 1$. Use integral calculus to find the surface area of the right circular cone obtained by revolving L about the y-axis.

$$y = 1 - x$$
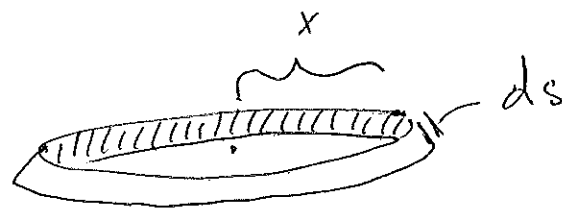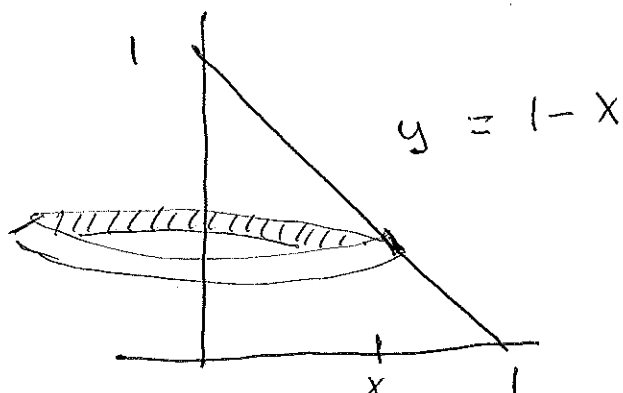
$$dS = 2\pi X \, ds$$

$$ds = \sqrt{(dx)^2 + (dy^2)} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$ds = \sqrt{1 + (-1)^2} \, dx = \sqrt{2} \, dx$$

$$dS = 2\sqrt{2}\, \pi X \, dx$$

$$S = 2\sqrt{2}\, \pi \int_0^1 x \, dx = 2\sqrt{2}\, \pi \, \frac{1}{2} \, x^2 \Big|_0^1$$

$$\boxed{S = \pi \sqrt{2}}$$

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Here are two problems (Problem 6 and Problem 6'). Choose *one of them* to solve, and solve it. (There is no "extra credit" for solving both.)

(15) **Problem 6.** Use L'Hôpital's rule to find $lim_{x\to\infty} \frac{x^{100}}{e^x}$. $\left(\frac{\infty}{\infty} \text{ form}\right)$

$$L = \lim_{x\to\infty} \frac{x^{100}}{e^x} = 100 \cdot \lim_{x\to\infty} \frac{x^{99}}{e^x} \quad \left(\text{still } \frac{\infty}{\infty}\right)$$

$$= 99 \cdot 100 \cdot \lim_{x\to\infty} \frac{x^{98}}{e^x} \quad \left(\frac{\infty}{\infty} \text{ still}\right)$$

$$= \cdots = 1 \cdot 2 \cdot 3 \cdots 99 \cdot 100 \cdot \lim_{x\to\infty} \frac{1}{e^x}$$

$$= (\text{An enormous number}) \cdot 0 = 0$$

(15) **Problem 6'.** Use the definition of *cosh x* and of *sinh x* to show that

$$2 \sinh x \cosh x = \sinh 2x.$$

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \quad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$2 \sinh x \cosh x = \frac{2}{4}\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right)$$

$$= \frac{2}{4}\left(e^{2x} - e^{-2x}\right)$$

$$= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) = \sinh(2x)$$

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