Math 221 Calculus 2
Professor John Maginnis

Midterm Exam 2
October 21, 2014

Your name: Solutions

Rec. Instr.: ______________________ Rec. Time: ______________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/9	/8	/9	/8
Problem	5	6	7	Total
Points	/8	/6	/12	/60

1. Find the arc length of the curve $y = \frac{1}{8}x^2 - \ln(x)$ for $1 \le x \le 2$.

$$\frac{dy}{dx} = \frac{1}{4}x - \frac{1}{x}, \text{ so } \left(\frac{dy}{dx}\right)^2 = \frac{1}{16}x^2 - \frac{1}{2} + \frac{1}{x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{16}x^2 + \frac{1}{2} + \frac{1}{x^2} = \left(\frac{1}{4}x + \frac{1}{x}\right)^2$$

$$s = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \frac{1}{4}x + \frac{1}{x}\, dx =$$

$$\left[\frac{1}{8}x^2 + \ln|x|\right]_1^2 = \frac{4}{8} + \ln(2) - \left(\frac{1}{8} + \ln(1)\right) = \boxed{\frac{3}{8} + \ln(2)}$$

2. Compute the sum of the series by writing it as a sum of two geometric series.

$$\sum_{n=1}^{\infty} \frac{5^{n-1} + 2^{n+3}}{10^n} = \sum_{n=1}^{\infty} \frac{5^{n-1}}{10^n} + \sum_{n=1}^{\infty} \frac{2^{n+3}}{10^n}$$

$$= \sum_{n=1}^{\infty} \frac{5^n \cdot \frac{1}{5}}{10^n} + \sum_{n=1}^{\infty} \frac{2^n \cdot 8}{10^n}$$

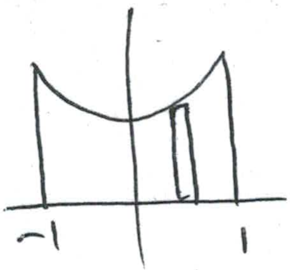
Note for $n=1$ we have $\frac{5^{n-1}}{10^n} = \frac{1}{10}$ and also $\frac{2^{n+3}}{10^n} = \frac{16}{10}$

$$= \sum_{n=1}^{\infty} \frac{1}{5}\left(\frac{5}{10}\right)^n + \sum_{n=1}^{\infty} 8 \cdot \left(\frac{2}{10}\right)^n$$

$$= \frac{1/10}{1 - 5/10} + \frac{16/10}{1 - 2/10}$$

$$= \frac{1}{5} + \frac{16}{8} = \boxed{2\frac{1}{5}} = \frac{11}{5}$$

3. Find the y-coordinate $\overline{y}$ of the centroid of the region under the parabola $y = x^2 + 1$ for $-1 \le x \le 1$. [Note $\overline{x} = 0$ by symmetry.]



$$M = A = \int_{-1}^{1} (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_{-1}^{1} =$$

$$\left(\frac{1}{3}+1\right) - \left(\frac{-1}{3}-1\right) = 2\tfrac{2}{3} = \boxed{\frac{8}{3}}$$

$$M_x = \frac{1}{2}\int_{-1}^{1} (x^2+1)^2\,dx = \frac{1}{2}\int_{-1}^{1} (x^4+2x^2+1)\,dx =$$

$$\frac{1}{2}\left[\frac{x^5}{5} + \frac{2}{3}x^3 + x\right]_{-1}^{1} = \frac{1}{2}\left[\frac{1}{5}+\frac{2}{3}+1-\left(\frac{-1}{5}-\frac{2}{3}-1\right)\right]$$

$$M_x = \boxed{\frac{28}{15}}. \text{ Thus } \overline{y} = \frac{M_x}{M} = \frac{28/15}{8/3} = \boxed{\frac{7}{10}}$$

4. Compute the second Taylor polynomial $T_2(x)$ at $a = \frac{\pi}{4}$ for the function $f(x) = \sec^2(x)$.

$$f'(x) = 2\sec(x)\cdot\sec(x)\tan(x) = 2\sec^2(x)\tan(x)$$

$$f''(x) = 4\sec(x)\big(\sec(x)\tan(x)\big)\tan x + 2\sec^2(x)\cdot\sec^2(x)$$

$$= 4\sec^2(x)\tan^2(x) + 2\sec^4(x)$$

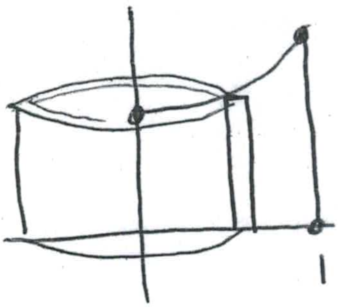
When $a = \pi/4$, $\sec(\pi/4) = \sqrt{2}$ and $\tan(\pi/4) = 1$.

$f(\pi/4) = 2$, $f'(\pi/4) = 4$, $f''(\pi/4) = 8+8 = 16$

$$T_2(x) = 2 + 4(x - \pi/4) + \frac{16}{2}(x-\pi/4)^2$$

$$\boxed{T_2(x) = 2 + 4(x-\pi/4) + 8(x-\pi/4)^2}$$

5. Find the volume of revolution formed by rotating the region in the first quadrant under the hyperbola $y^2 - x^2 = 1$ for $0 \le x \le 1$ around the y-axis.



$$dV = 2\pi R H \, dx$$

$$R = x, \quad H = y = \sqrt{x^2+1}$$

$$V = \int_0^1 2\pi x \sqrt{x^2+1} \, dx$$

$$u = x^2+1, \quad du = 2x\,dx$$

$$V = \int_1^2 \pi \sqrt{u}\, du = \pi \left[\frac{u^{3/2}}{3/2} \right]_1^2 = \boxed{\frac{2\pi}{3}\left(2\sqrt{2}-1\right)}$$

$$\left[\frac{2\pi}{3} (x^2+1)^{3/2} \right]_0^1$$

6. Determine whether the series converges. Explain.

$$\sum_{n=1}^{\infty} \frac{2n}{5n+7}$$ diverges by the Divergence Test

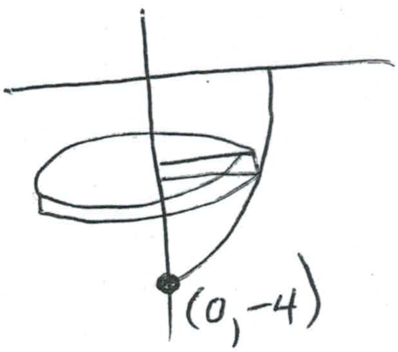
since $\displaystyle\lim_{n\to\infty} \left(\frac{2n}{5n+7} \right) = \boxed{\frac{2}{5}}$ is not zero.

Note $\displaystyle\lim_{n\to\infty} \left(\frac{2n}{5n+7} \right) = \lim_{n\to\infty} \left(\frac{2}{5+\frac{7}{n}} \right) = \frac{2}{5+0} = \frac{2}{5}$

or $\displaystyle\lim_{x\to\infty} \left(\frac{2x}{5x+7} \right) = \frac{\infty}{\infty} = \lim_{x\to\infty} \left(\frac{2}{5} \right) = \frac{2}{5}$

by L'Hôpital's Rule.

7. A tank full of water has the shape given by a volume of revolution, where the region in the fourth quadrant between the x-axis, y-axis, and the curve $y = x^4 - 4$ meters is rotated around the y-axis. Find the work required to pump out the water through a spout at the top of the tank (at $y = 0$). Leave your final answer in terms of g and ρ. Do <u>not</u> plug in $g = 9.8\frac{m}{sec^2}$ and $\rho = 1000\frac{kg}{m^3}$.



$$dV = \pi R^2 \, dy$$

$$R = x = \sqrt[4]{y+4}$$

$$R^2 = \sqrt{y+4}, \quad dV = \pi\sqrt{y+4}\,dy$$

$$dm = \rho\, dV, \quad dF = g\, dm, \quad dW = -y\, dF$$

(the distance to the spout is $0 - y = -y$)

$$W = \int_{-4}^{0} -\rho g \pi y \sqrt{y+4}\, dy$$

$$u = y+4, \quad du = dy, \quad y = u-4$$

$$W = \int_0^4 -\rho g \pi (u-4)\sqrt{u}\, du = -\rho g \pi \int_0^4 u^{3/2} - 4u^{1/2}\, du$$

$$W = -\rho g \pi \left[\frac{u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2}\right]_0^4 = -\rho g \pi \left(\frac{64}{5} - \frac{64}{3}\right)$$

$$W = -\rho g \pi (64)\left(\frac{-2}{15}\right) = \boxed{\frac{128}{15}\rho g \pi} \text{ Joules}$$