Math 221 Calculus 2
Professor John Maginnis

Final Exam
December 17, 2014

Your name: Solutions

Rec. Instr.: ______________________

Rec. Time: ______________________

Instructions:

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators.

For each test of convergence that you use, either give the name of the test, or briefly describe what the test says.

This exam is worth 120 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/9	/9	/8	/9
Problem	5	6	7	8
Points	/8	/8	/17	/9
Problem	9	10	11	12
Points	/9	/10	/8	/16

1. Determine whether the series converges or diverges. Explain.

$\sum_{n=1}^{\infty} \frac{n^2}{4^n}$ Use the Ratio Test:

$$\rho = \lim_{n \to \infty} \left| \frac{\frac{(n+1)^2}{4^{n+1}}}{\frac{n^2}{4^n}} \right| = \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{4^n}{4^{n+1}}$$

$$= (1)^2\left(\frac{1}{4}\right) = \boxed{\frac{1}{4}}$$

Since $\rho = \frac{1}{4} < 1$, the series converges.

2. Evaluate the indefinite integral.

$\int \frac{x+6}{x^2-4}\,dx$ $\qquad \frac{x+6}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$

$x+6 = A(x+2) + B(x-2)$ $\left\{ \begin{array}{l} \text{If } x=2,\ 8=4A \\ \text{If } x=-2,\ 4=-4B \end{array} \right\}$

$\left\{ \begin{array}{l} 1=A+B \\ 6=2A-2B \end{array} \right\}$ So $A=2$ and $B=-1$.

$$\int \frac{2}{x-2} - \frac{1}{x+2}\,dx =$$

$$\boxed{2\ln|x-2| - \ln|x+2| + C} = \ln\left|\frac{(x-2)^2}{x+2}\right| + C$$

3. Determine whether the series converges or diverges. Explain.

$$\sum_{n=1}^{\infty} \frac{n}{n^4+4}$$

Use the comparison test with

$$\frac{n}{n^4+4} < \frac{1}{n^3} \quad \left(\text{Note } n^4+4 > n^4 \text{ so } \frac{1}{n^4+4} < \frac{1}{n^4} \right)$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-series test with $p = 3 > 1$,

the series $\sum_{n=1}^{\infty} \frac{n}{n^4+4}$ also converges.

4. Find the Taylor series at $c = \frac{\pi}{4}$ for the function. You need to find a formula for the general term.

$$f(x) = \cos(2x)$$

$f(\pi/4) = \cos(\pi/2) = 0$

$f'(x) = -2\sin(2x) \qquad f'(\pi/4) = -2\sin(\pi/2) = -2$

$f''(x) = -4\cos(2x) \qquad f''(\pi/4) = 0$

$f'''(x) = 8\sin(2x) \qquad f'''(\pi/4) = 8$

$f^{(4)}(x) = 16\cos(2x) \qquad f^{(4)}(\pi/4) = 0$

$f^{(5)}(x) = -32\sin(2x) \qquad f^{(5)}(\pi/4) = -32$

There is a pattern, and $a_{2n} = 0$, $a_{2n+1} = \dfrac{(-1)^{n+1} 2^{2n+1}}{(2n+1)!}$

$$f(x) = \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n+1}}{(2n+1)!} \left(x - \frac{\pi}{4}\right)^{2n+1}$$

④ <u>Alternative method</u> :

$$\cos(2x) = \cos\left(2(x-\pi/4) + \pi/2\right)$$

$$= \cos\left(2(x-\pi/4)\right)\cos(\pi/2) - \sin\left(2(x-\pi/4)\right)\sin(\pi/2)$$

$$= -\sin\left(2(x-\pi/4)\right),$$ using the <u>addition formula for cosine</u>.

Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, we have

$$\cos(2x) = -\sum_{n=0}^{\infty} \frac{(-1)^n \left(2(x-\pi/4)\right)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n+1} (x-\pi/4)^{2n+1}}{(2n+1)!}$$

5. A spring hangs vertically. A mass of 10 kilograms is attached, and the spring is stretched by one meter. Find the work required in pulling the spring down one additional meter. You might want to use equations such as $F = ma$ and $F = -kx$. Also recall that $9.8 \frac{meters}{sec^2}$ is the acceleration of gravity.

If downward is positive, $F = ma = 10(9.8) = 98$ Newtons is the force of gravity. So the upward force of the spring is $-98 = -K(1)$, so $K = 98$, and $F = -98x$ is the force of the spring. So we must pull downward with force $F = 98x$.

The work $W = \int F\,dx = \int_1^2 98x\,dx = \left[49x^2\right]_1^2 =$

$49 \times 4 - 49 \times 1 = 49 \times 3 = \boxed{147 \text{ Joules}}$

6. Evaluate the indefinite integral.

$$\int \frac{dx}{(x^2+1)^{\frac{3}{2}}}$$

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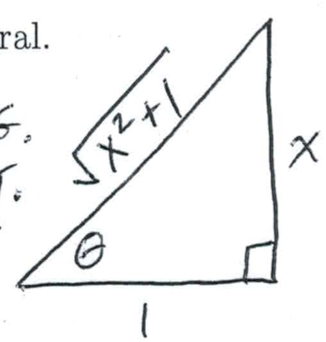


$x = \tan(\theta)$

$dx = \sec^2(\theta)\,d\theta$

$\sqrt{x^2+1} = \sec(\theta)$

$(x^2+1)^{3/2} = \sec^3(\theta)$

$$\int \frac{\sec^2(\theta)\,d\theta}{\sec^3(\theta)} = \int \frac{d\theta}{\sec(\theta)} = \int \cos(\theta)\,d\theta =$$

$$\sin(\theta) + C = \boxed{\frac{x}{\sqrt{x^2+1}} + C}$$

7. Consider the curve given by the parametrization.

$$x = \cos^3(t), \quad y = \sin^3(t)$$

(a) Compute the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 3\cos^2(t)(-\sin(t)) = -3\cos^2(t)\sin(t)$$

$$\frac{dy}{dt} = 3\sin^2(t)\cos(t)$$

(b) Find an equation of the tangent line to the curve when $t = \frac{\pi}{4}$.

Slope $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\sin^2(t)\cos(t)}{-3\cos^2(t)\sin(t)} = \frac{-\sin(t)}{\cos(t)}$

$\boxed{m=-1}$ $x_0 = \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$, $y_0 = \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{\sqrt{2}}{4}$

$\boxed{y - \frac{\sqrt{2}}{4} = -\left(x - \frac{\sqrt{2}}{4}\right)}$ or $\boxed{y = -x + \frac{\sqrt{2}}{2}}$

(c) Find the arc length of the curve for $0 \leq t \leq \frac{\pi}{2}$.

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)$$

$$= 9\sin^2(t)\cos^2(t)\left(\cos^2(t) + \sin^2(t)\right) = 9\sin^2(t)\cos^2(t)$$

Thus $ds = 3\sin(t)\cos(t)\,dt$, using $0 \leq t \leq \frac{\pi}{2}$.

$$s = \int_0^{\pi/2} 3\sin(t)\cos(t)\,dt = \int_0^1 3u\,du = \left[\frac{3}{2}u^2\right]_0^1$$

$u = \sin(t)$

$du = \cos(t)\,dt$

$$= \frac{3}{2} - 0 = \boxed{\frac{3}{2}}$$

8. Evaluate the indefinite integral. Integration by Parts (twice)

$$\int x^2 \sin(x)\, dx$$

$u = x^2$, $dv = \sin(x)\,dx$

$du = 2x\,dx$, $v = -\cos(x)$

$$= -x^2\cos(x) + \int +2x\cos(x)\,dx$$

$u = 2x$, $dv = \cos(x)\,dx$

$du = 2\,dx$, $v = \sin(x)$

$$= -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx$$

$$= \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$$

9. Determine whether the series converges or diverges. Explain.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$$

Leibniz Test for alternating series

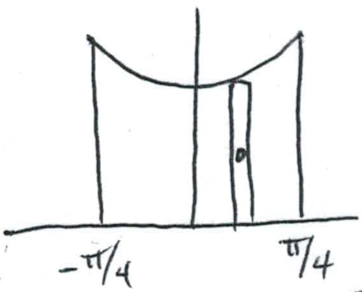
$$a_n = \frac{n}{n^2+1} > 0, \quad \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = \frac{0}{1+0} = \boxed{0}$$

If $f(x) = \dfrac{x}{x^2+1}$ then

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0 \text{ for } x \geq 1.$$

Thus $\{a_n\}$ is decreasing. Thus the series converges.

10. Find the y-coordinate $\overline{y}$ of the centroid of the region under the curve $y = \sec^2(x)$ for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$. Note that $\overline{x} = 0$ by symmetry.



$$M = A = \int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx = \Big[\tan(x)\Big]_{-\pi/4}^{\pi/4} = 1-(-1) = \boxed{2}$$

$$M_x = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^4(x)\,dx = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2(x)\left(1+\tan^2(x)\right)dx =$$

$$\frac{1}{2}\int_{-1}^{1} 1+u^2\,du = \frac{1}{2}\left[u+\frac{u^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{4}{3}-\left(\frac{-4}{3}\right)\right) = \boxed{4/3}$$

Pythagorean: $\sec^2(x) = 1+\tan^2(x)$

Substitution $u = \tan(x)$

$du = \sec^2(x)\,dx$

Then $\overline{y} = \dfrac{M_x}{M} = \dfrac{4/3}{2} = \boxed{\dfrac{2}{3}}$

11. Evaluate the limit. Show all work.

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$$

Use L'Hopital's Rule (twice)

$$= \frac{1-0-1}{0} = \frac{0}{0}, \quad \lim_{x\to 0} \frac{e^x-1}{2x} = \frac{1-1}{0} = \frac{0}{0},$$

$$\lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$$

An alternative method uses the Taylor polynomial $e^x \approx 1+x+\frac{x^2}{2}$, so

$e^x - x - 1 \approx \frac{x^2}{2}$, and $\displaystyle\lim_{x\to 0}\frac{e^x-x-1}{x^2} = \lim_{x\to 0}\frac{x^2/2}{x^2} = \boxed{\frac{1}{2}}$

12. Consider the curve given in polar coordinates, $r = 1 + 2\cos(2\theta)$.

(a) Solve for those θ where $r \le 0$.

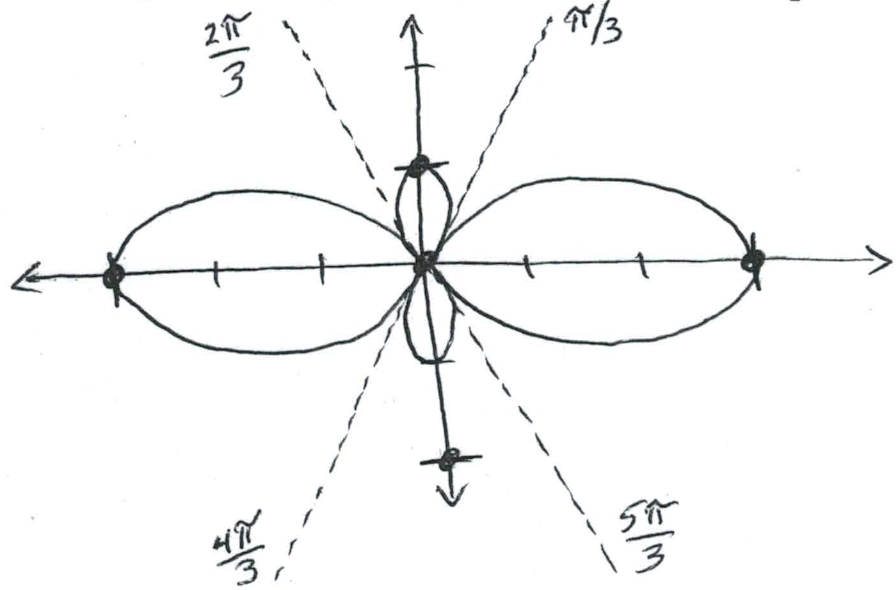
$$1 + 2\cos(2\theta) \le 0 \quad , \quad \cos(2\theta) \le -\frac{1}{2}$$

$$\frac{2\pi}{3} \le 2\theta \le \frac{4\pi}{3} \quad \text{or} \quad \frac{2\pi}{3} + 2\pi \le 2\theta \le \frac{4\pi}{3} + 2\pi$$

$$\boxed{\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}} \quad \text{or} \quad \boxed{\frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}}$$

(b) Graph the curve. In particular, plot the points where $\theta = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$.

θ	r
0	3
$\pi/2$	-1
π	3
$3\pi/2$	-1



(c) Find the total area (of the two small and two large loops) inside the curve.

$$A = \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos(2\theta)\right)^2 d\theta = \frac{1}{2}\int_0^{2\pi} \left(1 + 4\cos(2\theta) + 4\cos^2(2\theta)\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 + 4\cos(2\theta) + 4 \cdot \frac{1 + \cos(4\theta)}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(3 + 4\cos(2\theta) + 2\cos(4\theta)\right) d\theta$$

$$= \frac{1}{2}\left[3\theta + 2\sin(2\theta) + \frac{1}{2}\sin(4\theta)\right]_0^{2\pi} = \boxed{3\pi}$$