NAME Solutions

Rec. Instructor:

Signature

Rec. Time

CALCULUS II - EXAM 3
November 17, 2015

Show all work for full credit. No books, notes or calculators are permitted. The point value of each problem is given in the left-hand margin. You have 65 minutes.

Problem	Points	Points Possible	Problem	Points	Points Possible
1		10	6		10
2		8	7a		5
3a		6	7b		5
3b		6	8		6
3c		6	9a		6
4		6	9b		6
5a		6	10		8
5b		6			
			Total Score		100

(10) 1. Use the integral test to determine whether the following series converges or diverges. (The right answer is worth 2 points. Showing work is worth 8 points.)

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^{t} \frac{dx}{x\sqrt{\ln x}}$$

Let $u = \ln(x)$
$du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_2^t \frac{du}{\sqrt{u}} = \lim_{t\to\infty} 2u^{1/2} \Big|_2^t = \lim_{t\to\infty} 2\sqrt{\ln(x)} \Big|_2^t$$

$$= \lim_{t\to\infty} 2\sqrt{\ln(t)} - 2\sqrt{\ln(2)} = \infty,$$ since $\ln(t) \to \infty$ as $t \to \infty$.

Since integral diverges, so does the series.

(You might also note that the function $f(x) = \frac{1}{x\sqrt{\ln x}}$ is decreasing, so the integral test applies.)

(8) 2. Use the limit comparison test to determine whether the following series converges or diverges. (The answer is worth 2 points, showing work 6 points.)

$$\sum_{n=1}^{\infty} \frac{n^2+7}{3n^5-n^2}$$

Compare $\sum_{n=1}^{\infty} \frac{n^2}{3n^5} = \sum_{n=1}^{\infty} \frac{1}{3n^3} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^3}$

Note, the latter series is a p-series with $p = 3 > 1$, so it converges.

Apply limit comparison test:

$$\lim_{n\to\infty} \frac{n^2+7}{3n^5-n^2} \Big/ \frac{1}{3n^3} = \lim_{n\to\infty} \frac{n^2+7}{\cancel{n^2}(3n^3-1)} \cdot \frac{3n^{\cancel{3}}}{1}$$

$$= \lim_{n\to\infty} \frac{3(n^3+7n)}{3n^3-1} \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{3(1+7/n^2)}{3-1/n^3} = \frac{3}{3} = 1$$

Since the limit is a positive real number, the two series have the same behavior. Thus the original series converges.

3. Determine whether the following series converge or diverge. State clearly which test you are using and implement the test as clearly as you can. (The answer for each problem is worth 2 points and the work you show 4 points.)

(6) a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $= \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a p-series with $p = \frac{1}{2} < 1$, so the series diverges.

(6) b) $\sum_{n=1}^{\infty} \cos(1/n)$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1 \neq 0$$

Thus, the series diverges by divergence test.

(6) c) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

Use root test:

$$\rho = \lim_{n\to\infty} \left[\left(\frac{n}{2n+3}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+3} \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2+3/n} = \frac{1}{2} < 1$$

Thus, by root test, the series converges.

(6) 4. Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, and S_n denote the n-th partial sum. How many terms are required so that the error $|S - S_n|$ is less than $.01$, where S_n denotes the n-th partial sum?

Need $a_{n+1} < .01$, $\frac{1}{2(n+1)-1} < \frac{1}{100}$, $\frac{1}{2n+1} < \frac{1}{100}$

$\Leftrightarrow 2n+1 > 100$, $2n > 99$, $n > 99/2 = 49\frac{1}{2}$

Thus $n = 50$ is the minimal number of terms needed.

5. Determine whether the following series converge conditionally, converge absolutely, or diverge. (The answer is worth 2 points, the work 4 points.)

(6) a) $\sum_{n=2}^{\infty} \frac{\cos(n)}{n^{3/2}}$

$\sum_{n=2}^{\infty} \frac{|\cos(n)|}{n^{3/2}} \le \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} < \infty$, since the latter series is a p-series with $p = \frac{3}{2} > 1$.

Thus, by direct comparison test the original series converges absolutely.

(6) b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$, (diverges) p-series with $p = 1/2 < 1$.

The given series is an alternating series with terms satisfying $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, $\left\{\frac{1}{\sqrt{n}}\right\}$ is a decreasing sequence.

Thus, by Alternating Series test, the given series converges.

Therefore, the series converges <u>conditionally</u>.

(10) 6. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\,3^n}$. (Make clear the status of any end points.)

Start with ratio test: $\rho = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(n+1)\,3^{n+1}} \Big/ \frac{(x-3)^n}{n\,3^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n\,3^n}{(x-3)^n} \right| = |x-3| \cdot \lim_{n\to\infty} \frac{n}{(n+1)} \cdot \frac{1}{3}$$

$$= \frac{|x-3|}{3} \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{|x-3|}{3}$$

$\rho < 1 \iff \frac{|x-3|}{3} < 1 \iff |x-3| < 3$

0 3 6

Next, test endpts:

At $x = 0$, $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by Alt. series test, since $1/n$ decreases to 0.

$x = 6$, $\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, harmonic series.

Thus interval of convergence is $[0, 6)$

(5) 7. a) Use sigma notation to write down the general formula for the Taylor series of $f(x)$ centered at $x = c$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

(5) b) Apply the formula in part (a) to find the Maclaurin series ($c = 0$) for $f(x) = e^{2x}$. (Start by finding the n-th derivative of $f(x)$.)

$f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x}$

$\vdots$

$f^{(n)}(x) = 2^n e^{2x}$

Thus $f^{(n)}(0) = 2^n e^0 = 2^n$, for $n = 0, 1, 2, 3, \ldots$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

(valid for all $x \in \mathbb{R}$)

(6) 8. Given $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, find the terms up to x^5 in the Maclaurin series for $\ln(1+x^2)\sin x$.

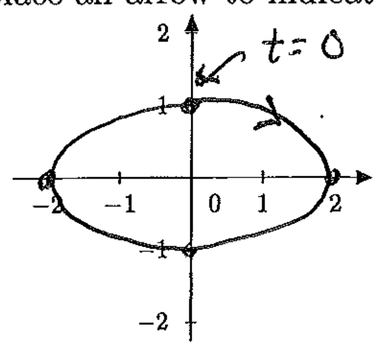
$$= \left(x^2 - \frac{x^4}{2} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= x^2\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) - \frac{x^4}{2}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right)$$

$$= x^3 - \frac{1}{6}x^5 + \cdots \quad - \frac{x^5}{2} + \cdots$$

$$= x^3 - \frac{2}{3}x^5 + \text{higher degree terms}$$

(6) 9. a) Sketch the graph of the curve given by the parametric equations $x = 2\sin t$, $y = \cos t$, $0 \le t \le 2\pi$. Indicate the starting point ($t = 0$) and place an arrow to indicate the direction traveled.



t	x	y
0	0	1
$\pi/2$	2	0
π	0	-1
$3\pi/2$	-2	0
2π	0	1

(6) b) Eliminate the parameter t to express the curve in part a) as an equation in terms of x and y.

$$\left(\frac{x}{2}\right)^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

$$\frac{x^2}{2^2} + y^2 = 1$$

(8) 10. A curve is given by the parametric equations $x = 2t - \sin t$, $y = 1 - \cos t$. Find the slope dy/dx of the curve at $t = \frac{\pi}{2}$.

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{\sin(t)}{2 - \cos(t)}$$

$$dy/dx \Big|_{t=\pi/2} = \frac{1}{2 - 0} = \frac{1}{2}$$