NAME Solutions Rec. Instructor: ____________

Signature ____________ Rec. Time ____________

CALCULUS II - FINAL EXAM
December 16, 2015, 6:20-8:10 p.m.

Show all work for full credit. No books, notes or calculators are permitted. The point value of each problem is given in the left-hand margin. You have 1 hour and 50 minutes.

Problem	Points	Points Possible	Problem	Points	Points Possible
1a		10	8		18
1b		12	9		11
2a		10	10		11
2b		12	11		10
3		8	12		6
4		14	13		12
5		16	14		18
6		10	15		10
7		12			
			Total Score		200

$\frac{d}{dx}\tan x = \sec^2 x$ $\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

$$\int \tan x \, dx = -\ln|\cos x| + C \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(\frac{x}{a}) + C \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$$

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x \, dx,$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx\,,$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x \, dx$$

$$\int \sqrt{a^2 - u^2} \, du = \frac{1}{2}\left(u\sqrt{a^2-u^2} + a^2 \sin^{-1}\frac{u}{a}\right) + C\,,$$

$$\int \sqrt{u^2 \pm a^2} \, du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$$

Units of force: pounds, newtons; Gravitational acceleration: $g = 9.8$ m/sec^2
Work = Force $\times$ Distance; Units of work: ft-lbs, newton-meters = joules;
Hooke's Law for springs: $F = kx$, where x is the distance stretched from rest position.

Moments: For the region between $y = f(x)$ and $y = g(x)$, with $a \leq x \leq b$,
$M_x = \frac{1}{2}\int_a^b f(x)^2 - g(x)^2 \, dx\,,$ $M_y = \int_a^b x(f(x) - g(x)) \, dx\,.$

Centroid and Center of Mass: $\overline{x} = M_y/M$, $\overline{y} = M_x/M$

Taylor Remainder: $|R_n(x)| \leq \frac{K}{(n+1)!}|x-a|^{n+1}$, with $K = \max_{a \leq c \leq x} |f^{(n+1)}(c)|$.

Maclaurin Series: $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

1. Evaluate the following integrals.

(10) a) $\displaystyle\int \frac{\ln(x)}{x^3}\,dx = \int \underbrace{x^{-3}}_{dv} \underbrace{\ln(x)}_{u}\,dx$

$u = \ln x \qquad dv = x^{-3}dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{x^{-2}}{-2}$

$$= uv - \int v\,du$$

$$(\ln x)\frac{x^{-2}}{-2} - \int \frac{x^{-2}}{-2}\frac{1}{x}\,dx$$

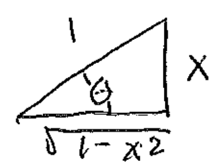
$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\frac{x^{-2}}{-2} + C = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$

(12) b) $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$



$\Rightarrow \cos\theta = \frac{\sqrt{1-x^2}}{1}$

$$= \int \frac{\sin^2\theta\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\sin^2\theta\cancel{\cos\theta}}{\cancel{\cos\theta}}\,d\theta$$

use reduction $n = 2$

$$= -\frac{\sin\theta\cos\theta}{2} + \frac{1}{2}\int 1\,d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C$$

$$= -\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) + C$$

2. Evaluate the following integrals.

(10) a) $\int \frac{e^x}{1+e^{2x}}\,dx$ Let $u = e^x$, $du = e^x\,dx$

$$= \int \frac{du}{1+u^2} = \tan^{-1}u + C = \tan^{-1}(e^x) + C$$

(12) b) $\int \frac{x+8}{x^3+4x}\,dx = \int \frac{x+8}{x(x^2+4)}\,dx$

$$\frac{x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow x+8 = A(x^2+4) + (Bx+C)x$$
$$x+8 = (A+B)x^2 + Cx + 4A$$

Equating x^2 terms: $A+B=0$
Equating x " : $C=1$
Equating constant terms: $4A=8 \Rightarrow A=2$
Then $B=-A=-2$

$$\int \frac{2}{x} + \frac{-2x+1}{x^2+4}\,dx = \int \frac{2}{x} - \frac{2x}{x^2+4} + \frac{1}{x^2+4}\,dx$$

$u = x^2+4$
$du = 2x\,dx$

$$= 2\ln|x| - \int\frac{du}{u} + \int\frac{dx}{x^2+2^2}$$

from cover sheet

$$= 2\ln|x| - \ln|x^2+4| + \frac{1}{2}\tan^{-1}(x/2) + C$$

(8) 3. Evaluate the improper integral or show that it diverges

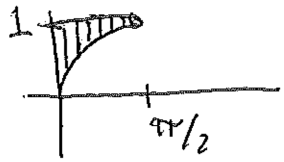
$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 (x-2)^{-\frac{1}{2}}\,dx = \lim_{t\to 2^+} 2(x-2)^{\frac{1}{2}}\Big|_t^5$

$$= \lim_{t\to 2^+} 2\sqrt{3} - 2\sqrt{t-2}$$

$$= 2\sqrt{3}$$

4. Let R be the region trapped between $y = 1$ and $y = \sin x$ with $0 \le x \le \pi/2$.

(6) a) Find the area of the region R.



$$A = \int_0^{\pi/2} 1 - \sin x \, dx = x + \cos x \Big|_0^{\pi/2}$$

$$= \pi/2 + \cos(\pi/2) - (0 + \cos 0)$$

$$= \pi/2 + 0 - 1 = \pi/2 - 1$$

(8) b) Find $\bar{y}$, the y coordinate of the centroid of R. (Do not calculate $\bar{x}$.) Hint: See cover sheet.

$$M_x = \frac{1}{2}\int_0^{\pi/2} 1^2 - \sin^2 x \, dx = \frac{1}{2}\left[x - \int_0^{\pi/2} \sin^2 x \, dx\right]_0^{\pi/2}, \text{ Use reduction } n=2$$

$$= \frac{1}{2}\left[x - \left[-\frac{\sin x \cos x}{2} + \frac{1}{2}x\right]\right]\Big|_0^{\pi/2}$$

$$= \frac{1}{4}x + \frac{1}{4}\sin x \cos x \Big|_0^{\pi/2} = \left(\frac{1}{4}\cdot\frac{\pi}{2} + \frac{1}{4}\cdot 1 \cdot 0\right) - (0 + 0)$$

$$= \pi/8 \qquad \bar{y} = M_x/A = \frac{\pi/8}{\pi/2 - 1}$$

5. Evaluate the following limits or indicate that they diverge. Show all work.

(8) a) $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{3x^2}$ $\frac{0}{0}$-type, so use L'Hopital

$$= \lim_{x \to 0} \frac{2e^{2x} - 2}{6x}, \ \frac{0}{0}\text{-type, again L'Hopital}$$

$$= \lim_{x \to 0} \frac{4e^{2x}}{6} = \frac{4e^0}{6} = 2/3$$

(8) b) $\lim_{x \to \infty} x^{2/x} = L$, ∞^0-type.

$$\ln L = \lim_{x \to \infty} \ln x^{2/x} = \lim_{x \to \infty} \frac{2}{x} \ln x = \lim_{x \to \infty} \frac{2 \ln x}{x} \leftarrow \frac{\infty}{\infty}\text{-type. Use L'Hopital}$$

$$= \lim_{x \to \infty} \frac{2/x}{1} = 0$$

Thus $L = e^0 = 1$

(10) 6. Solve the initial value problem, $\frac{dy}{dt} = \frac{\sec^2(t)}{e^y}$, $y(0) = 2$. Express your final answer in the form $y = f(t)$.

$$e^y\,dy = \sec^2 t\,dt$$

$$\int e^y\,dy = \int \sec^2 t\,dt$$

$$e^y = \tan t + C$$

when $t=0$, $y=2$, $\quad e^2 = \tan 0 + C = C$

$$e^y = \tan t + e^2$$

$$y = \ln\left(\tan t + e^2\right)$$

(12) 7. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-4)^n}{\sqrt{n}\,2^n}$. (Make clear the status of any end points.)

$$\rho = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{\sqrt{n+1}\cdot 2^{n+1}} \Big/ \frac{|x-4|^n}{\sqrt{n}\cdot 2^n} = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{|x-4|^n}\,\frac{\sqrt{n}}{\sqrt{n+1}}\,\frac{2^n}{2^{n+1}}$$

$$= |x-4| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \cdot \frac{1}{2} = \frac{|x-4|}{2} \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}} = \frac{|x-4|}{2}$$

Converges absolutely if $\frac{|x-4|}{2} < 1 \Leftrightarrow |x-4| < 2$

$\Leftrightarrow -2 < (x-4) < 2 \Leftrightarrow 2 < x < 6$

End pts: $x = 2$, $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}\cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, Alternating series with terms $\frac{1}{\sqrt{n}}$ decreasing to 0 as $n\to\infty$. Thus series converges by Alt. series test.

$x = 6$, $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}\cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series, $p = \frac{1}{2} < 1$, diverges.

Thus, the interval of convergence is $2 \leq x < 6$.

8. Determine whether the following series converge or diverge. State clearly which test you are using and implement the test as clearly as you can. The answer for each problem is worth 2 points and the work you show 4 points.

(6) a) $\sum_{n=3}^{\infty} \frac{1}{e^{1/n}}$

$$\lim_{n\to\infty} \frac{1}{e^{1/n}} = \frac{1}{e^0} = 1 \neq 0$$

Thus series diverges by divergence test,

(6) b) $\sum_{n=2}^{\infty} \frac{n}{n^3+n^2+2}$

$$< \sum_{n=2}^{\infty} \frac{n}{n^3} = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

The latter series is a p-series with $p=2$, so it converges. The original series converges by direct comparison test.

(6) c) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

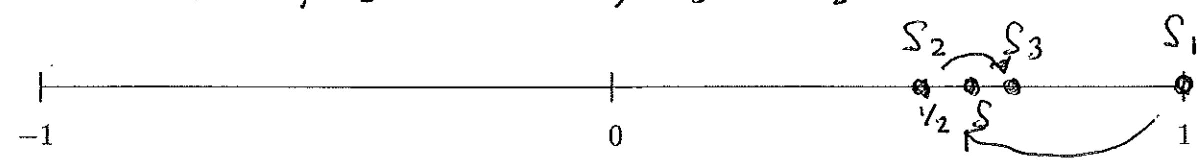
Use ratio test: $\rho = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$

$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$. Thus series converges by ratio test.

9. Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots$

(4) a) Mark the positions of the partial sums S_1, S_2, S_3 and the sum S on the number line below.

$S_1 = 1$, $S_2 = 1 - \frac{1}{2} = \frac{1}{2}$, $S_3 = 1 - \frac{1}{2} + \frac{1}{6} = 2/3$



(4) b) How many terms are required to estimate the sum S with an error less than .01?

$|S - S_n| < a_{n+1} = \frac{1}{(n+1)!}$. Need $\frac{1}{(n+1)!} < \frac{1}{100}$, $(n+1)! > 100$

Note: $4! = 24$, $5! = 120$. $n = 4$ will suffice

(3) c) Evaluate the sum S by making use of a familiar series; see cover page.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \cancel{1-1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$

$$\boxed{S = 1 - e^{-1}}$$

10. Let $T_2(x)$ be the second degree Taylor polynomial for the function $f(x) = \cos(2x)$ centered at $x = \pi/2$.

(8) a) Calculate $T_2(x)$.

$f(x) = \cos(2x)$, $\quad f(\pi/2) = \cos(\pi) = -1$

$f'(x) = -2\sin(2x)$, $\quad f'(\pi/2) = -2\sin(\pi) = 0$

$f''(x) = -4\cos(2x)$, $\quad f''(\pi/2) = -4\cos(\pi) = 4$

$$T_2(x) = f(\pi/2) + f'(\pi/2)(x-\pi/2) + \tfrac{1}{2} f''(\pi/2)(x-\pi/2)^2$$
$$= -1 + 0 + 2\,(x-\pi/2)^2$$
$$= -1 + 2\,(x-\pi/2)^2$$

(3) b) $T_2(x)$ is the unique quadratic polynomial satisfying what three properties in terms of the graph of $f(x)$. (These are the defining properties of a Taylor polynomial.)

$T_2(\pi/2) = f(\pi/2)$

$T_2'(\pi/2) = f'(\pi/2)$, same slope

$T_2''(\pi/2) = f''(\pi/2)$, same concavity

(4) 11. a) Use the geometric series formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find the Maclaurin series for $\dfrac{1}{1+x^2}$,

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

(4) b) By integrating your expansion in part a) obtain the Maclaurin series for $\tan^{-1} x$.

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

At $x=0$, $0 = \tan^{-1}(0) = \sum_{n=0}^{\infty} 0 + C$, so $C = 0$, and we get

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

(2) c) Use part (b) to evaluate the sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots =$ $\tan^{-1}(1) = \pi/4$

valid, since series converges by A.S.T.

(6) 12. Use the series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$, and $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$, to find the terms up to x^4 for the Maclaurin series of $e^{x^2} \ln(1+x^2)$.

$$e^{x^2} \ln(1+x^2) = \left(1 + x^2 + \frac{x^4}{2} + \cdots\right)\left(x^2 - \frac{1}{2}x^4 + \cdots\right)$$

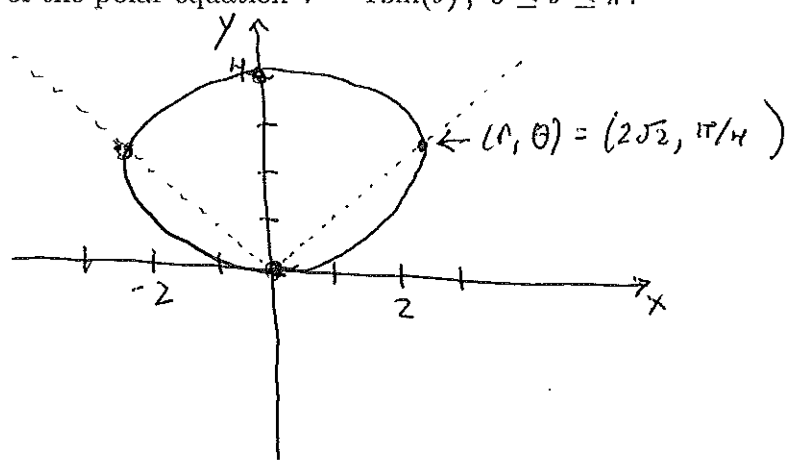
$$= (1+x^2)(x^2 - \tfrac{1}{2}x^4) + \cdots \leftarrow \text{terms exceeding } x^4$$

$$= x^2 + x^4 - \tfrac{1}{2}x^4 + \cdots$$

$$= x^2 + \tfrac{1}{2}x^4 + \cdots$$

(6) 13. a) Sketch the graph of the polar equation $r = 4\sin(\theta)$, $0 \le \theta \le \pi$.

θ	$r = 4\sin\theta$
0	0
$\pi/4$	$4 \cdot \sqrt{2}/2 = 2\sqrt{2} \approx 2.8$
$\pi/2$	4
$\frac{3}{4}\pi$	$2\sqrt{2}$
π	0



(6) b) Convert the polar equation in part a) to a rectangular equation in x and y, and state what familiar shape it is?

$$r = 4\sin\theta$$

$$r^2 = 4r\sin\theta$$

$$x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y + \_ = 0$$

$$x^2 + (y-2)^2 = 4$$

circle of radius 2 centered at $(0, 2)$

14. Consider the curve with parametric equations $x = e^{3t}$, $y = \sin(2t)$.

(6) a) Find the slope of the curve at $t = 0$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(2t)}{3e^{3t}}, \quad \text{At } t = 0, \quad \left.\frac{dy}{dx}\right|_{t=0} = \frac{2\cos(0)}{3e^0} = \frac{2}{3}$$

(4) b) Find the equation of the tangent line to the curve at $t = 0$.

$$x = e^0 = 1, \; y = \sin(0) = 0, \; m = 2/3$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 2/3\,(x - 1), \qquad y = \frac{2}{3}x - 2/3$$

(8) c) Set up **but do not evaluate** an integral representing the length of the curve $x = e^{3t}$, $y = \sin(2t)$, $0 \leq t \leq 2$.

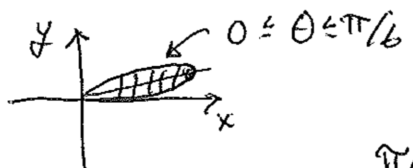
$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(3e^{3t})^2 + (2\cos(2t))^2}\, dt$$

$$= \int_0^2 \sqrt{9e^{6t} + 4\cos^2(2t)}\, dt$$

(10) 15. Set up **but do not evaluate** an integral representing the area bounded by one petal of the rose $r = \sin(6\theta)$.

$6\theta = \pi/2 \Rightarrow \theta = \pi/12$

θ	r
0	0
$\pi/12$	$\sin(\pi/2) = 1$
$\pi/6$	$\sin(\pi) = 0$



$$A = \frac{1}{2}\int_a^b r^2\, d\theta = \frac{1}{2}\int_0^{\pi/6} \sin^2(6\theta)\, d\theta$$