Math 221 Name ______________________

Final Exam Part 1 July 29, 2015

Page 1	Page 2	Page 3	Page 4	Page 5	Total

You must show all work to get full credit. No notes or calculators are allowed on this exam.

1. (10 pts) Evaluate: $\int (3x+7)e^{3x}\,dx$

2. (10 pts) Evaluate: $\int \sin^4(x)\cos^5(x)\,dx$

3. (10 pts) Evaluate: $\int \frac{dx}{x^2\sqrt{x^2-9}}$

4. (10 pts) Use partial fraction decomposition to evaluate $\int \frac{3-x}{x(x^2+1)}\, dx$

5. (10 pts) Find the centroid of the region bounded by $y = x$, $y = x^2$, $x = 0$, and $x = 1$. Note: $y = x$ is above $y = x^2$ for x in $(0, 1)$.

6. (5 pts) Find the derivative of the parametric curve $(t^2 + 1, t^3 - 4t)$ at the point $t = 1$.

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{x}{2} - \frac{1}{2}\sin x\cos x + C$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{x}{2} + \frac{1}{2}\sin x\cos x + C$$

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1} x\cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2} x\,dx$$

$$\int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1} x\cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x\cos^n x\,dx$$

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \tan^m x\,dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x\,dx$$

$$\int \cot x\,dx = -\ln|\csc x| + C = \ln|\sin x| + C$$

$$\int \cot^m x\,dx = -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x\,dx$$

$$\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$$

$$\int \sec^m x\,dx = \frac{\tan x\sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx$$

$$\int \csc x\,dx = \ln\left|\csc x - \cot x\right| + C$$

$$\int \csc^m x\,dx = -\frac{\cot x\csc^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \csc^{m-2} x\,dx$$

$$\int \sin mx\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n)$$

$$\int \sin mx\cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C \quad (m \neq \pm n)$$

$$\int \cos mx\cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n)$$

$W = \int_a^b F(x)\,dx$

Hooke's Law for stretching/compressing: F(x)=kx

Work against gravity:

$W = \int_a^b L(y)\,dy$

Where $L(y) = g \times \text{density} \times A(y) \times \text{vertical distance}$

$s = \int_a^b \sqrt{1 + f'(x)^2}\,dx$

$A = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}$

$F = \rho g \int_a^b \text{depth} \times \text{area of strip}$

$x_{CM} = \frac{M_y}{M}$

$y_{CM} = \frac{M_x}{M}$

$M_y = \rho \int_a^b x(f_1(x) - f_2(x)\,dx$

$M_x = \frac{1}{2}\rho \int_a^b (f_1(x)^2 - f_2(x)^2)\,dx$

$M = \rho \int_a^b (f_1(x) - f_2(x))\,dx$

- The nth *Taylor polynomial* centered at $x = a$ for the function $f(x)$ is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

When $a = 0$, $T_n(x)$ is also called the nth *Maclaurin polynomial*.

- If $f^{(n+1)}(x)$ exists and is continuous, then we have the *error bound*

$$|T_n(x) - f(x)| \le K \frac{|x-a|^{n+1}}{(n+1)!}$$

where K is a number such that $|f^{(n+1)}(u)| \le K$ for all u between a and x.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$; $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$; $A = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$

$x = r\cos(\theta),\ y = r\sin(\theta),\ r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x}(\text{if } x \neq 0)$

Area $=\frac{1}{2}\int_\alpha^\beta f(\theta)^2\, d\theta$; Arc Length $= \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

Geometric series starting at n=k with ratio r: $\frac{cr^k}{1-r}$

Function $f(x)$	**Maclaurin series**	**Converges to $f(x)$ for**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$	$\|x\| < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots$	$\|x\| < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\|x\| < 1$ and $x = 1$
$\tan^{-1} x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\|x\| < 1$ and $x = 1$
$(1+x)^a$	$\sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$	$\|x\| < 1$

Math 221 Name ____________________
Final Exam Part 2 July 29, 2015

Page 1	Page 2	Page 3	Page 4	Page 5	Page 6	Page 7	Total

You must show all work to get full credit. No notes or calculators are allowed on this exam.

Directions: For the problems 1 through 10, write true if the statement is correct and if the stament is wrong, write false and edit the statement so that it is correct.

1. (2 pts) Let $\{a_n\}$ be a sequence that is bounded above. Then $\{a_n\}$ converges.

2. (2 pts) Let $\sum_{n=1}^{\infty} a_n$ be a series where $\{a_n\}$ is a sequence that does not converge to 0. Then $\sum_{n=1}^{\infty} a_n$ converges.

3. (2 pts) Let $\sum_{n=1}^{\infty} a_n$ be a series where $a_n = f(n)$ and $f(x)$ is a continuous, decreasing function with $f(x) > 0$, for all x and $\int_1^{\infty} f(x)\, dx$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges.

4. (2 pts) Let $\sum_{n=1}^{\infty}(-1)^n a_n$ be a series where $\{a_n\}$ is a positive, decreasing sequence. Then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

5. (2 pts) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series and suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge by the limit comparison test.

6. (2 pts) Let $\sum_{n=1}^{\infty} a_n$ be a series such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ where $L < 1$. Then $\sum_{n=1}^{\infty} a_n$ converges conditionally.

7. (2 pts) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be infinite series with $0 \leq a_n \leq b_n$ for all n. Suppose $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges by the comparison test.

8. (2 pts) Let $S_N = \sum_{n=1}^{N} a_n$ be the sequence of partial sums. Suppose $\lim_{N\to\infty} S_N$ exists. Then $\sum_{n=1}^{\infty} a_n$ may diverge.

9. (2 pts) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p \geq 1$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

10. (2 pts) Let $\sum_{n=1}^{\infty} a_n$ be a series such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$. Then $\sum_{n=1}^{\infty} a_n$ diverges.

11. (10 pts) Does $\sum_{n=5}^{\infty} \frac{\ln(n)}{n}$ converge conditionally, absolutely, or diverge?

12. (15 pts) Find the interval of convergence for $\sum_{n=2}^{\infty} \frac{n}{n^2+1}(x-1)^n$.

13. (15 pts) Find a power series for $\frac{1}{(1+x)^2}$. Hint: Use the Maclaurin Series of $\frac{1}{1-x}$.

14. (10 pts) Find a Taylor Series centered at c=-1 for the function

$$f(x) = x^2 + 2x + 2 \ .$$

15. (5 pts) **Bonus:** Show that the following equality is true using Maclaurin series:

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{x}{2} - \frac{1}{2}\sin x\cos x + C$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{x}{2} + \frac{1}{2}\sin x\cos x + C$$

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1} x\cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2} x\,dx$$

$$\int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1} x\cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x\cos^n x\,dx$$

$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \tan^m x\,dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x\,dx$$

$$\int \cot x\,dx = -\ln|\csc x| + C = \ln|\sin x| + C$$

$$\int \cot^m x\,dx = -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x\,dx$$

$$\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$$

$$\int \sec^m x\,dx = \frac{\tan x\sec^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx$$

$$\int \csc x\,dx = \ln\left|\csc x - \cot x\right| + C$$

$$\int \csc^m x\,dx = -\frac{\cot x\csc^{m-2} x}{m-1} + \frac{m-2}{m-1}\int \csc^{m-2} x\,dx$$

$$\int \sin mx\sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n)$$

$$\int \sin mx\cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} + C \quad (m \neq \pm n)$$

$$\int \cos mx\cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C \quad (m \neq \pm n)$$

$W = \int_a^b F(x)\,dx$

Hooke's Law for stretching/compressing: F(x)=kx

Work against gravity:

$W = \int_a^b L(y)\,dy$

Where $L(y) = g \times \text{density} \times A(y) \times \text{vertical distance}$

$s = \int_a^b \sqrt{1+f'(x)^2}\,dx$

$A = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}$

$F = \rho g\int_a^b \text{depth} \times \text{area of strip}$

$x_{CM} = \frac{M_y}{M}$

$y_{CM} = \frac{M_x}{M}$

$M_y = \rho\int_a^b x(f_1(x) - f_2(x)\,dx$

$M_x = \frac{1}{2}\rho\int_a^b (f_1(x)^2 - f_2(x)^2)\,dx$

$M = \rho\int_a^b (f_1(x) - f_2(x))\,dx$

- The nth *Taylor polynomial* centered at $x = a$ for the function $f(x)$ is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

When $a = 0$, $T_n(x)$ is also called the nth *Maclaurin polynomial*.

- If $f^{(n+1)}(x)$ exists and is continuous, then we have the *error bound*

$$|T_n(x) - f(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$$

where K is a number such that $|f^{(n+1)}(u)| \leq K$ for all u between a and x.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$; $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$; $A = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$

$x = r\cos(\theta),\ y = r\sin(\theta),\ r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x}(\text{if } x \neq 0)$

Area $= \frac{1}{2}\int_\alpha^\beta f(\theta)^2\, d\theta$; Arc Length $= \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

Geometric series starting at n=k with ratio r: $\frac{cr^k}{1-r}$

Function $f(x)$	**Maclaurin series**	**Converges to** $f(x)$ **for**
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$	$\|x\| < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots$	$\|x\| < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\|x\| < 1$ and $x = 1$
$\tan^{-1} x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\|x\| < 1$ and $x = 1$
$(1+x)^a$	$\sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$	$\|x\| < 1$