Math 221 Calculus 2
Professor John Maginnis

Midterm Exam 2
October 17, 2017

Your name: Solutions

Rec. Instr.: ____________ Rec. Time: ____________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

| Problem | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Points | /8 | /6 | /12 | /8 |
| Problem | 5 | 6 | | Total |
| Points | /8 | /18 | | /60 |

1. Find the arc length of the curve.

$$y = \frac{1}{3}x^{\frac{3}{2}} = \frac{\sqrt{x^3}}{3} \quad \text{for} \quad 0 \leq x \leq 4$$

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{3}{2} x^{1/2} = \frac{\sqrt{x}}{2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x}{4}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x}{4}}$$

$$s = \int_0^4 \sqrt{1+\frac{x}{4}}\,dx$$

Substitution $u = 1 + \frac{x}{4}$

$$du = \frac{1}{4}\,dx$$

$$4\,du = dx$$

$$s = \int_1^2 \sqrt{u}\,(4\,du) = \left[4\,\frac{u^{3/2}}{3/2}\right]_1^2 = \boxed{\frac{8}{3}\left(2\sqrt{2}-1\right)} = \frac{16\sqrt{2}}{3} - \frac{8}{3}$$

2. A force of 3 pounds will stretch a spring six inches from its equilibrium or natural length. Find the work (in foot-pounds) required to stretch the spring from its natural length to a length of six inches beyond its natural length.

$\boxed{F = Kx}$ (or could use $F = -Kx$), Hooke's Law

3 pounds $= K\left(\frac{1}{2}\text{ feet}\right)$ since 6 inches $= \frac{1}{2}$ foot

$\boxed{K = 6}$

$$W = \int F\,dx = \int_0^{1/2} 6x\,dx = \left[3x^2\right]_0^{1/2}$$

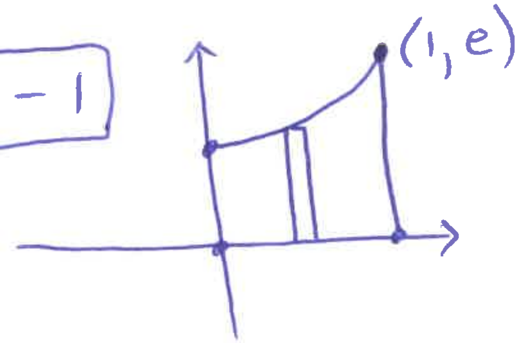
$$= \frac{3}{4} - 0$$

$$= \boxed{\frac{3}{4}\text{ foot-pounds}}$$

3. Find the centroid of the region under $y = e^x$ for $0 \leq x \leq 1$.

(a) Find the area bounded by $y = e^x$, $x = 0$, $x = 1$, and $y = 0$.

$$M = A = \int_0^1 e^x \, dx = \left[ e^x \right]_0^1 = \boxed{e - 1}$$




(b) Find the moment $M_x$ with respect to the $x$-axis.

$$M_x = \frac{1}{2} \int_0^1 (e^x)^2 \, dx = \frac{1}{2} \int_0^1 e^{2x} \, dx = \left[ \frac{e^{2x}}{4} \right]_0^1$$

$$= \boxed{\frac{e^2 - 1}{4}}$$

(c) Find the moment $M_y$ with respect to the $y$-axis.

$$M_y = \int_0^1 x e^x \, dx = \left[ x e^x \right]_0^1 - \int_0^1 e^x \, dx =$$

Parts $u = x$, $dv = e^x dx$
$du = dx$, $v = e^x$

$$\left[ x e^x - e^x \right]_0^1 =$$

$$(e - e) - (0 - 1) = \boxed{1}$$

(d) Compute the centroid $(\overline{x}, \overline{y})$.

$$\overline{x} = \frac{M_y}{M} = \boxed{\frac{1}{e - 1}}, \quad \overline{y} = \frac{M_x}{M} = \boxed{\frac{\frac{e^2 - 1}{4}}{e - 1}} = \boxed{\frac{e + 1}{4}}$$

4. Determine whether the sequence converges (compute a limit).

$$a_n = \frac{\ln(n^2+1)}{n}$$

$$\lim_{n\to\infty} \frac{\ln(n^2+1)}{n} = \lim_{x\to\infty} \frac{\ln(x^2+1)}{x} = \text{(L'Hôpital's Rule)}$$

$$\lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{1} = \lim_{x\to\infty} \frac{2/x}{1+1/x^2} = \frac{0}{1+0} = \boxed{0.}$$

The sequence converges to zero.

5. (a) Write out the first three terms $a_1, a_2$ and $a_3$ for the sequence.

$$a_n = \frac{3^{n+1}}{2^{2n-1}}$$

$$a_1 = \frac{3^2}{2^1} = \frac{9}{2}, \quad a_2 = \frac{3^3}{2^3} = \frac{27}{8}, \quad a_3 = \frac{3^4}{2^5} = \frac{81}{32}$$

(b) Evaluate the sum of the geometric series.

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{2n-1}} = \frac{a_1}{1-r} \quad \text{where } a_1 = \boxed{\frac{9}{2}} \text{ and}$$

$$r = \frac{27/8}{9/2} = \boxed{\frac{3}{4}} = \frac{81/32}{27/8} \quad \left(\text{also } \frac{3^{n+1}}{2^{2n-1}} = 6\left(\frac{3}{4}\right)^n.\right)$$

$$\frac{9/2}{1-3/4} = \frac{9/2}{1/4} = \boxed{18}$$

6. Determine whether the series converges; list each test of convergence used.

(a)

$$\sum_{n=1}^{\infty} \frac{7n-1}{4n+3}$$ diverges by the Divergence Test (nth term test)

since $\lim_{n\to\infty} \frac{7n-1}{4n+3} = \lim_{n\to\infty} \frac{7-1/n}{4+3/n} = \frac{7-0}{4+0} = \boxed{\frac{7}{4}}$

and $\frac{7}{4} \neq 0$.

(b)

$$\sum_{n=1}^{\infty} \frac{7n-1}{4n^2+3}$$ diverges by the Limit Comparison Test

by comparing to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{7n-1}{4n^2+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{7n^2-n}{4n^2+3} = \lim_{n\to\infty} \frac{7-1/n}{4+3/n^2} = \boxed{\frac{7}{4}}$$

and $\frac{7}{4} \neq 0$.

(c)

$$\sum_{n=1}^{\infty} \frac{7n-1}{4n^3+3}$$ converges by the Comparison Test

(or can use the Limit Comparison Test again)

$$\frac{7n-1}{4n^3+3} < \frac{7n}{4n^3+3} < \frac{7n}{4n^3} = \frac{7}{4n^2}.$$ The p-series test

with $p = 2 > 1$ shows $\sum_{n=1}^{\infty} \frac{7}{4n^2} = \frac{7}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.