Math 221 Calculus 2
Professor John Maginnis

Midterm Exam 3
November 14, 2017

Your name: Solutions

Rec. Instr.: ____________ Rec. Time: ____________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3
Points	/10	/10	/12
Problem	4	5	Total
Points	/16	/12	/60

1. Determine whether the series converges or diverges.

   List each test of convergence used.

$$\sum_{n=1}^{\infty} \left(\frac{4n^3 - 3}{5n^3 + 7} \right)^n$$

This series converges by the Root Test:

$$\lim_{n \to \infty} \sqrt[n]{\left(\frac{4n^3 - 3}{5n^3 + 7} \right)^n} = \lim_{n \to \infty} \frac{4n^3 - 3}{5n^3 + 7} =$$

$$\lim_{n \to \infty} \frac{4 - 3/n^3}{5 + 7/n^3} = \frac{4 - 0}{5 + 0} = \boxed{\frac{4}{5}}$$

(or could use L'Hôpital's Rule)

and $\frac{4}{5} < 1$.

2. Use the power series for $\frac{1}{1-x}$ (a geometric series) to find the Taylor series centered at $a = 0$ for the function below.

$$f(x) = \frac{4x^2}{9 - x^3}$$

Recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$

Then $\frac{4x^2}{9-x^3} = \left(\frac{4x^2}{9}\right)\left(\frac{1}{1-\frac{x^3}{9}}\right)$

$$= \frac{4x^2}{9} \sum_{n=0}^{\infty} \left(\frac{x^3}{9}\right)^n$$

$$= \frac{4x^2}{9} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{4x^{3n+2}}{9^{n+1}}}$$

3. Find the second Taylor polynomial $p_2(x)$ centered at $a = -\frac{\pi}{4}$ for the function

$f(x) = \tan(x)$.

$$f'(x) = \sec^2(x)$$

$$f''(x) = 2\sec(x)\left(\sec(x)\tan(x)\right)$$
$$= 2\sec^2(x)\tan(x)$$

Then evaluate at $a = -\pi/4$

$$f\left(-\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right) = -1$$
$$f'\left(-\frac{\pi}{4}\right) = \sec^2\left(-\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2$$
$$f''\left(-\frac{\pi}{4}\right) = 2 \cdot 2 \cdot (-1) = -4$$

So $p_2(x) = -1 + 2\left(x+\frac{\pi}{4}\right) - \frac{4}{2}\left(x+\frac{\pi}{4}\right)^2$

$$\boxed{p_2(x) = -1 + 2\left(x+\frac{\pi}{4}\right) - 2\left(x+\frac{\pi}{4}\right)^2}$$

4. Find the interval of convergence for the power series.

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 3^n}$$

Start with the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(x+2)^{n+1}}{(n+1)3^{n+1}}}{\frac{(x+2)^n}{n\cdot 3^n}} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{(x+2)^{n+1}}{(x+2)^n} \right| =$$

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{|x+2|}{3}\right) = \left(\frac{|x+2|}{3}\right) \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x+2|}{3}.$$

Then $\boxed{\frac{|x+2|}{3} < 1}$ for $|x+2| < 3$, or $-3 < x+2 < 3$.

The power series converges absolutely for $\boxed{-5 < x < 1.}$

Check endpoints: If $x = 1$, $\sum_{n=1}^{\infty} \frac{3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is the Divergent harmonic series (p-series test with $p=1$).

If $x = -5$, $\sum_{n=1}^{\infty} \frac{(-3)^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

converges by the Alternating Series Test (note $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$). The interval of convergence is $\boxed{-5 \le x < 1.}$

5. A curve is given by the parametrization $x = t^2 + 1, \quad y = 4t^5 + t^4$.

(a) Find an equation of the tangent line to this curve at the point when $t = 1$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{20t^4 + 4t^3}{2t} = \boxed{10t^3 + 2t^2}$$

When $t = 1$, the slope $m = 10(1)^3 + 2(1)^2 = \boxed{12.}$

When $t = 1$, $x = 2$ and $y = 5$.

$\boxed{y - 5 = 12(x - 2)}$ or $y = 12x - 19$.

(b) Find the concavity $\frac{d^2y}{dx^2}$ of the curve at the point when $t = 1$.

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{dy}{dx}\right)/dt}{dx/dt} = \frac{30t^2 + 4t}{2t}$$

$\frac{d^2y}{dx^2} = 15t + 2$. When $t = 1$, the concavity is $15(1) + 2 = \boxed{17.}$