Math 221 Calculus 2
Professor John Maginnis

Final Exam
December 13, 2017

Your name: Solutions

Rec. Instr.: ________

Rec. Time: ________

Instructions:

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 120 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4	5
Points	/10	/10	/10	/10	/20
Problem	6	7	8	9	10
Points	/10	/10	/20	/10	/10

1. Find $\overline{y}$, the y-coordinate of the centroid of the region under the curve $y = 1 - x^4$ for $-1 \leq x \leq 1$. Note that $\overline{x} = 0$ by symmetry.

$$A = \int_{-1}^{1} (1-x^4)\,dx = \left[x - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{5} + \frac{4}{5} = \boxed{\frac{8}{5}}$$

$$M_x = \frac{1}{2}\int_{-1}^{1} (1-x^4)^2\,dx = \frac{1}{2}\int_{-1}^{1} (1-2x^4+x^8)\,dx$$

$$= \frac{1}{2}\left[x - \frac{2}{5}x^5 + \frac{1}{9}x^9\right]_{-1}^{1} = \frac{1}{2}\left(1-\frac{2}{5}+\frac{1}{9}\right) - \frac{1}{2}\left(-1+\frac{2}{5}-\frac{1}{9}\right)$$

$$= \frac{45-18+5}{45} = \boxed{\frac{32}{45}}$$

$$\overline{y} = \frac{M_x}{A} = \frac{32/45}{8/5} = \boxed{\frac{4}{9}}$$

2. Find an equation of the tangent line to the curve given by the parametrization, at the point when $t = 2$.

$x = 6t - t^2$ and $y = 6t - t^3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6-3t^2}{6-2t}, \quad m = \frac{6-12}{6-4} = \frac{-6}{2} = -3.$$

When $t=2$, $x = 12-4 = 8$ and $y = 12-8 = 4$.

$$\boxed{y-4 = -3(x-8)} \quad \text{or} \quad y = -3x+28.$$

3. Find the area inside the curve given in polar coordinates.

$$r = \pi\theta - \theta^2 \quad \text{for} \quad 0 \le \theta \le \pi$$

$$A = \int \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} \frac{1}{2}(\pi\theta - \theta^2)^2 \, d\theta =$$

$$\frac{1}{2}\int_0^{\pi} (\pi^2\theta^2 - 2\pi\theta^3 + \theta^4) \, d\theta = \frac{1}{2}\left[\frac{\pi^2}{3}\theta^3 - \frac{\pi\theta^4}{2} + \frac{\theta^5}{5}\right]_0^{\pi}$$

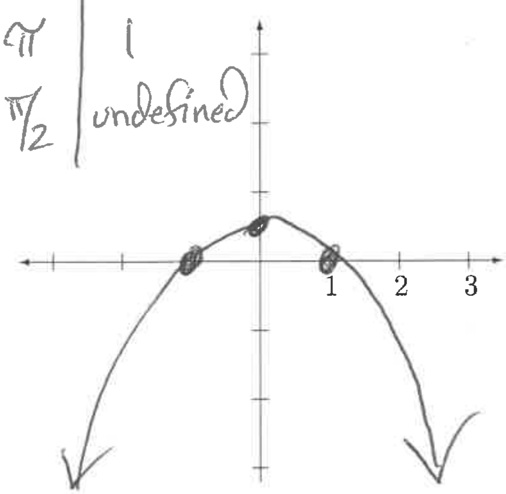
$$= \frac{1}{2}\left[\frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5}\right] = \frac{\pi^5}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= \boxed{\frac{\pi^5}{60}}$$

4. Rewrite the following polar equation in Cartesian coordinates, describe the curve, and give a graph of the curve.

$$r = \frac{1}{1 + \sin(\theta)}$$

θ	r
0	1
$\pi/2$	$1/2$
π	1
$3\pi/2$	undefined



$$r(1 + \sin\theta) = 1$$

$$r + r\sin\theta = 1$$

$$r + y = 1$$

$$r = 1 - y$$

$$x^2 + y^2 = r^2 = 1 - 2y + y^2$$

$$\boxed{x^2 = 1 - 2y}$$

This is a parabola, facing down, with vertex $x = 0$, $y = 1/2$

$$\left(y = \frac{1}{2} - \frac{1}{2}x^2\right)$$

5. For the series

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$$

(a) Use the Ratio Test to show this series converges.

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n\cdot 2^n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} \frac{n}{n+1} = \boxed{\frac{1}{2}}$$

and since $\boxed{\frac{1}{2} < 1}$, the series converges.

(b) Use the Comparison Test to show this series converges.

For $n \geq 1$, we have $n \cdot 2^n \geq 2^n$, so that

$$\boxed{\frac{1}{n\cdot 2^n} \leq \frac{1}{2^n}.}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series with ratio $\frac{1}{2}$.

(c) Use the Limit Comparison Test to show this series converges.

$$\lim_{n\to\infty} \frac{\frac{1}{n 2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{n 2^n} = \lim_{n\to\infty} \frac{1}{n} = \boxed{0.}$$

Thus $\frac{1}{n 2^n}$ is eventually much smaller than $\frac{1}{2^n}$ (a convergent geometric series.)

6. Find the arc length of the curve given by the parametrization.

$x = 12\, t^{\frac{5}{2}}$ and $y = 15t^2$ for $0 \le t \le 1$

$\frac{dx}{dt} = \left(\frac{5}{2}\right)\left(12 t^{3/2}\right) = 30\, t^{3/2}$, $\frac{dy}{dt} = 30t$

$s = \int_0^1 \sqrt{900t^3 + 900t^2}\, dt = \boxed{\int_0^1 30t\sqrt{t+1}\, dt}$

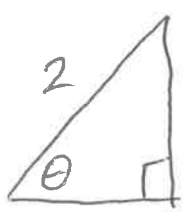
Substitution $u = t+1$, $du = dt$, and $t = u - 1$.

$\int_1^2 30(u-1)\sqrt{u}\, du = \int_1^2 30u^{3/2} - 30u^{1/2}\, du =$

$\left[12u^{5/2} - 20u^{3/2}\right]_1^2 = 48\sqrt{2} - 40\sqrt{2} - (12-20) = \boxed{8\sqrt{2} + 8.}$

7. Use a trigonometric substitution to evaluate the indefinite integral.

$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\, dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}(2\cos\theta)\, d\theta =$$



$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$\int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = \int \tan^2\theta\, d\theta =$

$\int(-1 + \sec^2\theta)\, d\theta =$

$-\theta + \tan\theta + C = \boxed{-\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{\sqrt{4-x^2}} + C}$

8. Use the remainder estimate for the integral test, $|S - S_N| \leq \int_N^\infty f(x)\,dx$, to estimate the error in using the fifth partial sum S_5 as an approximation for the series.

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

The error $\left|S - S_5\right| \leq \int_5^\infty \frac{dx}{x^2+x} = \lim_{R\to\infty} \int_5^R \frac{dx}{x^2+x}$.

Partial fractions $x^2+x = x(x+1)$

$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$, $1 = A(x+1) + Bx$

If $x=0$, get $A=1$

If $x=-1$, get $B=-1$.

$$\int_5^R \frac{dx}{x^2+x} = \int_5^R \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \Big[\ln|x| - \ln|x+1|\Big]_5^R$$

$$= \left[\ln\left|\frac{x}{x+1}\right|\right]_5^R = \ln\left(\frac{R}{R+1}\right) - \ln\left(\frac{5}{6}\right)$$

$$\lim_{R\to\infty}\left(\ln\left(\frac{R}{R+1}\right) - \ln\left(\frac{5}{6}\right)\right) = \ln(1) - \ln\left(\frac{5}{6}\right)$$

$$= \boxed{-\ln\left(\frac{5}{6}\right)} \text{ or } \boxed{\ln\left(\frac{6}{5}\right)} \text{ or } \boxed{\ln 6 - \ln 5.}$$

9. Use the Taylor polynomial $p_3(x) = x - \frac{x^3}{6}$ for the sine function to help approximate the following definite integral.

$$\int_0^1 x^7 \sin\left(\frac{x^4}{10}\right)\, dx \approx \int_0^1 \left(\frac{x^{11}}{10} - \frac{x^{19}}{6{,}000}\right) dx =$$

$$\left[\frac{x^{12}}{120} - \frac{x^{20}}{120{,}000}\right]_0^1 =$$

$$\frac{1}{120} - \frac{1}{120{,}000} =$$

$$\frac{1000-1}{120{,}000} =$$

$$\boxed{\frac{999}{120{,}000}} = \frac{333}{40{,}000} = .008325$$

Approximations:

$$\sin(x) \approx x - \frac{x^3}{6}$$

$$\sin\left(\frac{x^4}{10}\right) \approx \frac{x^4}{10} - \frac{x^{12}}{6000}$$

$$x^7 \sin\left(\frac{x^4}{10}\right) \approx \frac{x^{11}}{10} - \frac{x^{19}}{6000}$$

10. Assume that the partial sums S_N for a series $\sum_{n=1}^{\infty} a_n$ satisfy $S_N = \frac{2N}{N+1}$. Compute the limit of the sequence a_n.

Note $\boxed{\lim_{n\to\infty} a_n = 0}$ by the Divergence Test since

$\lim_{N\to\infty} \frac{2N}{N+1} = \boxed{2}$, so the series converges to 2.

Or use $a_n = S_n - S_{n-1} = \frac{2n}{n+1} - \frac{2(n-1)}{n} = \frac{2n^2 - 2(n^2-1)}{n(n+1)}$

$\boxed{a_n = \frac{2}{n^2+n}}$, so that $\lim_{n\to\infty}\left(\frac{2}{n^2+n}\right) = \boxed{0.}$