Math 221 Calculus 2
Professor John Maginnis

Midterm Exam 3
April 4, 2017

Your name: Solutions

Rec. Instr.: ______________ Rec. Time: ______________

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/4	/10	/12	/12
Problem	5	6		Total
Points	/12	/10		/60

1. Determine whether the sequence converges (compute a limit).

$$a_n = \frac{2n^2 - 1}{3n^2 + 5}$$

$$\boxed{\lim_{n \to \infty} \frac{2n^2 - 1}{3n^2 + 5} = \frac{2}{3} \text{ converges}}$$

(Use $\dfrac{2 - \frac{1}{n^2}}{3 + \frac{5}{n^2}}$, or L'Hopital $\dfrac{4n}{6n}$, or ratio of leading coefficients)

2. Determine whether the series converges; list each test of convergence used.

(a)

$$\sum_{n=0}^{\infty} \frac{3^n}{n!}$$ converges by the Ratio Test:

$$\lim_{n \to \infty} \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = \lim_{n \to \infty} \left(\frac{3^{n+1}}{3^n}\right) \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{3}{n+1} = \boxed{0}$$

Since $0 < 1$, the series converges.

(b)

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{2n}\right)^{3n}$$ converges by the Root Test:

$$\lim_{n \to \infty} \sqrt[n]{\left| \left(\frac{n+1}{2n}\right)^{3n} \right|} = \lim_{n \to \infty} \left(\frac{n+1}{2n}\right)^3 = \left(\frac{1}{2}\right)^3 = \boxed{\frac{1}{8}}.$$

Since $\frac{1}{8} < 1$, the series converges.

3. Determine whether the series converges; list each test of convergence used.

(a)

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$$ converges by the Absolute Convergence Test

and the comparison test $\left|\frac{\sin(n)}{n^3}\right| \le \frac{1}{n^3}$

since $-1 \le \sin(n) \le 1$,

and the p-series test with $p = 3 > 1$.

(b)

$$\sum_{n=2}^{\infty} \frac{2^n}{e^n - 3}$$ converges by the Limit Comparison Test.

Remark: $\frac{2^n}{e^n - 3} > \left(\frac{2}{e}\right)^n$ is not useful (cannot use the Comparison Test).

$$\lim_{n\to\infty} \frac{\frac{2^n}{e^n-3}}{\frac{2^n}{e^n}} = \lim_{n\to\infty} \frac{e^n}{e^n-3} = \lim_{n\to\infty} \frac{1}{1-\frac{3}{e^n}} = \frac{1}{1-0} = \boxed{1}.$$

Note that $\sum_{n=2}^{\infty} \frac{2^n}{e^n}$ is a convergent geometric series with ratio $r = \frac{2}{e} < 1$.

4. Determine whether the series converges; list each test of convergence used.

(a)

$$\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$$ diverges by the Comparison Test:

For $n \geq 3$, $\ln(n) > 1$, so that $\boxed{\frac{\ln(n)}{n} > \frac{1}{n}}$.

Note $\sum_{n=3}^{\infty} \frac{1}{n}$ is a divergent harmonic series

(diverges by the p-series test with $p=1$)

(could also use the integral test, with $\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_3^t = \infty$)

(b)

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)^n$$ diverges by the Divergence Test

since $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \boxed{\frac{1}{e} \neq 0.}$

If $y = \left(1-\frac{1}{x}\right)^x$, then $\ln(y) = x\ln\left(1-\frac{1}{x}\right) = \dfrac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}$.

$$\lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{x}}\left(\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-1}{1-\frac{1}{x}} = \boxed{-1}$$

(L'Hôpital)

Then $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1-\frac{1}{x}\right)} = e^{-1} = \boxed{\frac{1}{e}}$.

5. Find the interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \text{ converges for } \boxed{-1 \le x < 1}$$

<u>Ratio Test</u>: $\lim_{n\to\infty} \left| \dfrac{\frac{x^{n+1}}{\sqrt{n+1}}}{\frac{x^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \sqrt{\frac{n}{n+1}} \cdot \frac{x^{n+1}}{x^n} \right|$

(could use the Root Test)

$= |x| < 1, \; -1 < x < 1.$

<u>Endpoints</u>: If $x = 1$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ <u>diverges</u> by the <u>p-series test</u> with $p = \frac{1}{2} \le 1$.

If $x = -1$, then $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ <u>converges</u> by the <u>Alternating Series Test</u>. Note that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

and $a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$ (since $n+1 > n$).

6. (a) Use the remainder estimate for alternating series to find the number N such that the series is approximated by the partial sum S_N with accuracy within $.001 = \frac{1}{1000}$.

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}} \qquad a_n = |b_n| = \frac{1}{n^{3/2}}$$

$$\boxed{|S - S_N| \le a_{N+1}} = \frac{1}{(N+1)^{3/2}} = .001$$

$$(N+1)^{3/2} = 1000, \quad \sqrt{N+1} = 10, \quad N+1 = 100,$$

$$\boxed{N = 99}$$

(b) Use the remainder estimate for the integral test to find the number N such that the series is approximated by the partial sum S_N with accuracy within $.001 = \frac{1}{1000}$.

$$S = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \qquad f(x) = \frac{1}{x^{3/2}} = x^{-3/2}$$

$$\boxed{|S - S_N| \le \int_N^{\infty} \frac{dx}{x^{3/2}}} = \lim_{t \to \infty} \left[\frac{x^{-1/2}}{-1/2} \right]_N^t =$$

$$\lim_{t \to \infty} \left(\frac{-2}{\sqrt{t}} + \frac{2}{\sqrt{N}} \right) = 0 + \frac{2}{\sqrt{N}} = \frac{2}{\sqrt{N}} = .001$$

$$\sqrt{N} = 2000, \quad \boxed{N = 4{,}000{,}000}$$