Math 221 Calculus 2 Final Exam
Professor John Maginnis May 10, 2017

Your name: Solutions

Rec. Instr.: ________________

Rec. Time: ________________

Instructions:

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 120 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4	5
Points	/12	/7	/7	/6	/5
Problem	6	7	8	9	10
Points	/12	/5	/7	/7	/7
Problem	11	12	13	14	Total
Points	/12	/13	/10	/10	/120

1. Evaluate the integrals.

(a)

$$\int x^3 \ln(x)\, dx$$

Integration by Parts

$u = \ln(x)$ $\qquad dv = x^3\, dx$

$du = \dfrac{dx}{x}$ $\qquad v = \dfrac{1}{4}x^4$

$$\frac{1}{4}x^4 \ln(x) - \int \frac{1}{4}x^3\, dx = \boxed{\frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + C}$$

(b)

$$\int \frac{6x+6}{x^2-9}\, dx$$

Partial Fractions

$x^2 - 9 = (x-3)(x+3)$

$$\frac{6x+6}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}, \quad 6x+6 = A(x+3) + B(x-3)$$

If $x = 3$, $24 = 6A$, $A = 4$. If $x = -3$, $-12 = -6B$, $B = 2$.

$$\int \left(\frac{4}{x-3} + \frac{2}{x+3}\right) dx = \boxed{4\ln|x-3| + 2\ln|x+3| + C}$$

2. Find the Taylor polynomial $T_2(x)$ of degree two for the function $y = \frac{\ln(x)}{x}$ at the point $a = 1$.

When $a=1$, $y = \frac{\ln(1)}{1} = \boxed{0.}$

Then $\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln(x)}{x^2} = \frac{1-\ln x}{x^2}\bigg|_{a=1} = \boxed{1}$

$$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1-\ln x)(2x)}{x^4} = \frac{2\ln(x) - 3}{x^3}\bigg|_{a=1} = \boxed{-3}$$

$$\boxed{T_2(x) = (x-1) - \frac{3}{2}(x-1)^2}$$

3. Find the Taylor series at $a = 0$ for the function $y = \frac{x^3}{x^2-5}$.

$$\frac{x^3}{x^2-5} = \left(-\frac{x^3}{5}\right) \cdot \frac{1}{1-\left(\frac{x^2}{5}\right)}$$ = using the formula for a geometric series

$$-\frac{x^3}{5}\sum_{n=0}^{\infty}\left(\frac{x^2}{5}\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{-x^{2n+3}}{5^{n+1}}}$$

4. Determine whether the series converges or diverges; list each test of convergence used.

$$\sum_{n=1}^{\infty} \frac{n}{2n^2-1}$$

This series Diverges by the Comparison Test since

$2n^2-1 < 2n^2$, and so $\boxed{\frac{n}{2n^2-1} > \frac{n}{2n^2} > \frac{1}{2n}.}$

Note $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is one half of the divergent harmonic series (the p-series test with $p=1$)

(or use limit comparison $\displaystyle\lim_{n\to\infty} \frac{n/2n^2-1}{1/n} = \frac{1}{2}$)

5. Use the remainder estimate for the alternating series test to find the number N such that the series is approximated by the partial sum S_N with accuracy within $.1 = \frac{1}{10}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

$$|S-S_N| \leq \frac{1}{\sqrt[3]{N+1}} \leq \frac{1}{10}$$

$$\sqrt[3]{N+1} \geq 10$$

$$N+1 \geq 1000$$

$$N \geq \boxed{999}$$

6. A curve is described using the parameter t by $x = t^2 - 2t$ and $y = t^3$.

(a) Find the slope $\frac{dy}{dx}$ of the tangent line to this curve when $t = 2$.

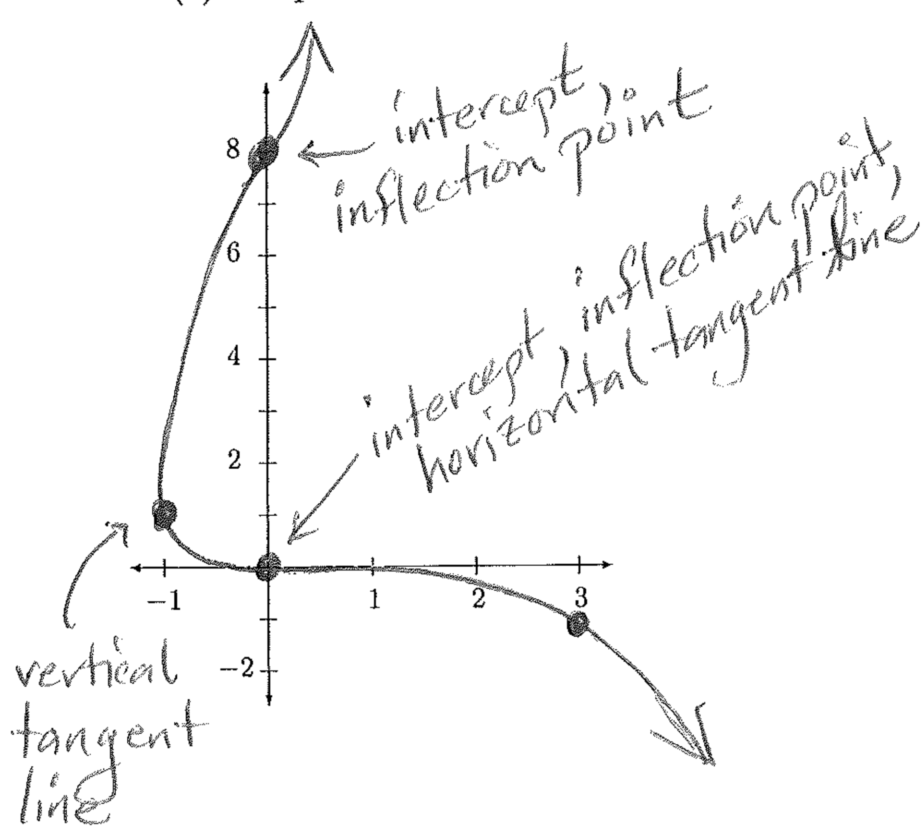
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t-2}\Bigg|_{t=2} = \frac{12}{4-2} = \boxed{6}$$

(b) Find the concavity $\frac{d^2y}{dx^2}$ of this curve when $t = 2$.

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)/dt}{dx/dt} = \frac{(2t-2)(6t)-(3t^2)(2)}{(2t-2)^3}$$

$$= \frac{6t^2-12t}{(2t-2)^3}\Bigg|_{t=2} = \frac{24-24}{8} = \boxed{0}$$

(c) Graph the curve for $-1 \le t \le 2$.



t	x	y
-1	3	-1
0	0	0
1	-1	1
2	0	8

7. Determine whether the sequence converges or diverges (compute a limit).

$$a_n = \frac{\ln(n)}{\sqrt{n}}$$

This sequence converges to zero.

$$\boxed{\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}}} = \lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \boxed{0}$$

L'Hopital's Rule

8. Determine whether the series converges or diverges; list each test of convergence used.

$$\sum_{n=0}^{\infty} \frac{n!}{(2n+1)!}$$

This series converges by the Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(2n+3)!}}{\frac{n!}{(2n+1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+3)} = \lim_{n\to\infty} \frac{n+1}{4n^2+10n+6}$$

$$= \lim_{n\to\infty} \frac{1/n + 1/n^2}{4 + 10/n + 6/n^2} = \frac{0+0}{4+0+0} = \boxed{0}$$

$$\left(\text{or use L'Hôpital } \lim_{n\to\infty} \frac{1}{8n+10} \right)$$

Since $0 < 1$, the series converges.

9. Evaluate the integral of the hyperbolic function.

$$\int \tanh^3(x)\,\text{sech}^5(\text{x})\,\text{dx}$$

Substitution $u = \text{sech}(x)$

$du = -\text{sech}(x)\tanh(x)\,dx$

$\tanh^2(x) = 1 - \text{sech}^2(x) = 1 - u^2$

$$\int -(1-u^2)u^4\,du = \int -u^4 + u^6\,du =$$

$$-\frac{1}{5}u^5 + \frac{1}{7}u^7 + C = \boxed{-\frac{1}{5}\text{sech}^5(x) + \frac{1}{7}\text{sech}^7(x) + C}$$

10. Find the arc length of the parametrized curve $x = 2t^2$, $y = t^3$ for $0 \leq \text{t} \leq 1$.

$$\frac{dx}{dt} = 4t, \quad \frac{dy}{dt} = 3t^2, \quad \frac{ds}{dt} = \sqrt{16t^2 + 9t^4}$$

$$\int_0^1 \sqrt{16t^2 + 9t^4}\,dt = \int_0^1 t\sqrt{16 + 9t^2}\,dt$$

$u = 16 + 9t^2$, $du = 18t\,dt$

$$\int_{16}^{25} \frac{1}{18}\sqrt{u}\,du = \left[\frac{1}{18}\frac{u^{3/2}}{3/2}\right]_{16}^{25} = \frac{1}{27}(125 - 64) = \boxed{\frac{61}{27}}$$

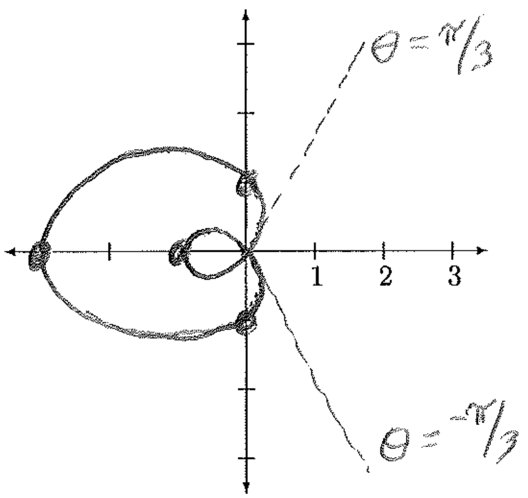
11. A curve is given in polar coordinates by $r = 1 - 2\cos(\theta)$.

(a) Find the angles θ where $r = 0$.

$1 - 2\cos(\theta) = 0$, $\cos(\theta) = 1/2$, $\theta = \boxed{\pm \pi/3}$

(or $\frac{\pi}{3}$ and $\frac{5\pi}{3}$)

(b) Graph the curve, plotting the points where $\theta = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$.



θ	$r = 1 - 2\cos\theta$
0	-1
$\pi/2$	1
π	3
$3\pi/2$	1

(c) Find the area inside the small loop.

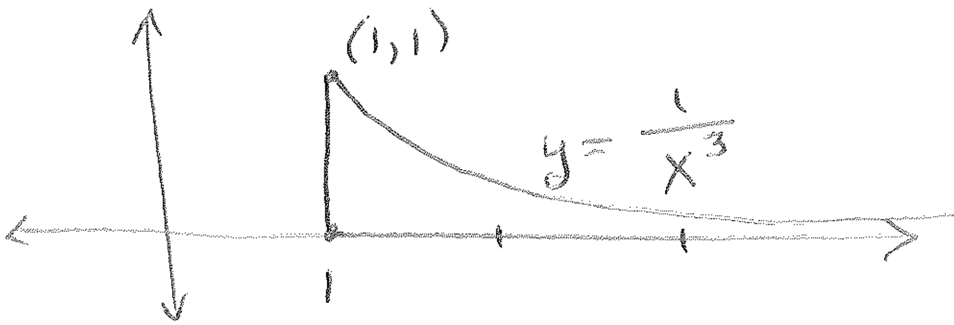
$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} (1 - 2\cos(\theta))^2 \, d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} 1 - 4\cos\theta + 4\cos^2\theta \, d\theta$$

$$= \left[\frac{\theta}{2} - 2\sin\theta + \theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/3}^{\pi/3} =$$

$$\left(\frac{\pi}{2} - 2\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right) - \left(\frac{-\pi}{2} - 2\left(\frac{-\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right)\right)$$

$$= \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = \boxed{\pi - \frac{3\sqrt{3}}{2}}$$

12. Find the centroid of the unbounded region under the curve $y = \frac{1}{x^3}$ for $x \geq 1$. Write each improper integral as a limit.



$$A = M = \int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^t$$

$$= \lim_{t\to\infty} \left(\frac{-1}{2t^2} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

$$M_y = \int_1^\infty x\left(\frac{1}{x^3}\right)dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx =$$

$$\lim_{t\to\infty} \left[\frac{-1}{x}\right]_1^t = \lim_{t\to\infty} \left(\frac{-1}{t} + 1\right) = \boxed{1}$$

$$M_x = \frac{1}{2}\int_1^\infty \left(\frac{1}{x^3}\right)^2 dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x^6}\,dx =$$

$$\lim_{t\to\infty} \left(\frac{-1}{10t^5} + \frac{1}{10}\right) = \boxed{\frac{1}{10}}$$

So $\bar{x} = 1/\tfrac{1}{2} = \boxed{2}$ and $\bar{y} = \dfrac{1/10}{1/2} = \boxed{\dfrac{1}{5}}$

13. Evaluate the integral.

$$\int \frac{dx}{x^2\sqrt{x^2+9}}$$

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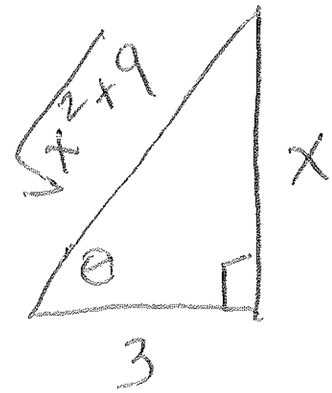


$x = 3\tan(\theta)$

$dx = 3\sec^2(\theta)\,d\theta$

$\sqrt{x^2+9} = 3\sec(\theta)$

$$\int \frac{3\sec^2(\theta)\,d\theta}{9\tan^2\theta \cdot 3\sec(\theta)} = \boxed{\frac{1}{9}\int \frac{\sec\theta\,d\theta}{\tan^2\theta}}$$

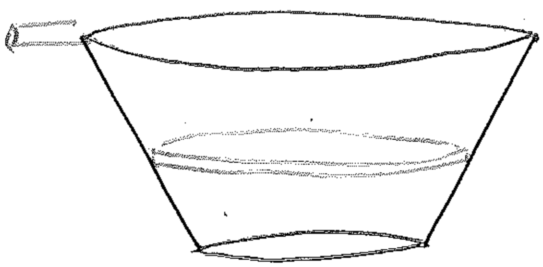
$$= \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta$$

Substitution $u = \sin(\theta)$, $du = \cos(\theta)\,d\theta$

$$\frac{1}{9}\int \frac{du}{u^2} = \frac{1}{9}\left(\frac{u^{-1}}{-1}\right) + C = \frac{-1}{9u} + C = \frac{-1}{9\sin\theta} + C$$

$$= \boxed{\frac{-\sqrt{x^2+9}}{9x} + C}$$

14. Find the work required to pump the water out of a full tank, one meter tall, pumped through a pipe at the top of the tank. Each horizontal cross section is circular, with radius $R = y + 1$ at height y for $0 \le y \le 1$. Write your answer in terms of g and ρ; do not plug in the numbers $g \doteq 9.8$ meters per second squared (the acceleration of gravity) and $\rho = 1000$ kilograms per cubic meter (the density of water).



Volume of horizontal slice is

$$dV = \pi R^2 dy = \pi (y+1)^2 dy$$

Mass $dM = \rho\, dV = \pi\rho (y+1)^2 dy$

Force (weight) $dF = g \cdot dM = \pi\rho g (y+1)^2 dy$

Work $dW = (1-y)\, dF = \pi\rho g (1-y)(y+1)^2 dy$

$$\int_0^1 \pi\rho g (1-y)(y^2+2y+1)\, dy = \int_0^1 \pi\rho g (1+y-y^2-y^3)\, dy$$

$$= \pi\rho g \left(y + \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4}\right)\Big]_0^1 = \pi\rho g \left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)$$

$$= \boxed{\frac{11}{12}\pi\rho g} \text{ Joules.}$$