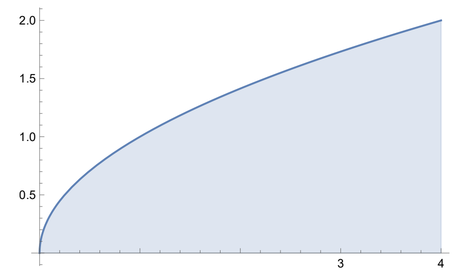
1. (a) (6 points) Draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the x-axis.

$$f(x) = \sqrt{x}, \quad x = 0, \quad x = 4 \quad \text{and} \quad y = 0.$$

Solution:



$$V = \pi \int_0^4 (f(x))^2 \, dx = \pi \int_0^4 x \, dx = \pi \left. \frac{x^2}{2} \right|_0^4 = \boxed{8\pi}$$

(b) (6 points) Use shell method to find the volume generated when the region between the curves

$$y = \sqrt{x}, \quad y = x^2 \quad \text{rotated around y-axis}$$

Solution: These curves intersect at $x = 0, 1$. The volume is thus

$$V = \int_0^1 x(\sqrt{x} - x^2) \, dx = \int_0^1 x^{3/2} - x^3 \, dx$$
$$= \left. \frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right|_0^1 = \frac{2}{5} - \frac{1}{4} = \boxed{\frac{3}{20}}$$

2. (a) (6 points) Find the arclength of the function $f(x) = \frac{4}{3}x^{3/2}$ where $0 \le x \le 1$.

Solution:

$$\begin{aligned} f' &= \frac{4}{3}\cdot\frac{3}{2}x^{1/2} = 2x^{1/2} \\ (f')^2 &= 4x \\ L &= \int_0^1 \sqrt{1+(f')^2}\,\mathrm{d}x \\ &= \int_0^1 \sqrt{1+4x}\,\mathrm{d}x \\ &= \frac{1}{4}\frac{2}{3}(1+4x)^{3/2}\Big|_0^1 \\ &= \boxed{\frac{1}{6}(5^{3/2}-1)} \end{aligned}$$

(b) (6 points) Find the surface area of the volume generated by revolving the curve $y = x^2$ from $(1, 1)$ to $(3, 9)$ around the y-axis.

Solution:

$$\begin{aligned} SA &= \int 2\pi x\,\mathrm{d}s \\ &= \int_1^3 2\pi x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\ &= \int_1^3 2\pi x\sqrt{1+4x^2}\,\mathrm{d}x \qquad (u = 1+4x^2; \mathrm{d}u = 8x\,\mathrm{d}x) \\ &= \frac{\pi}{4}\int_5^{37}\sqrt{u}\,\mathrm{d}u \\ &= \frac{\pi}{4}\frac{2}{3}\,u^{3/2}\Big|_5^{37} \\ &= \boxed{\frac{\pi}{6}(37^{3/2}-5^{3/2})} \end{aligned}$$

3. (12 points) Find the work required to pump all the water out of a cylinder that has a circular base of radius 5 ft and height 200 ft. Use the fact that the density of water is 62 lb/ft^3.

Solution:

$$\begin{aligned}
\text{force} &= \text{vol} \cdot \text{density} \\
&= 25\pi \, \mathrm{d}y \cdot 62 \\
\text{Work} &= \int \text{force} \cdot \text{distance} \\
&= \int_0^{200} 25\pi \cdot 62 \cdot (200 - y) \, \mathrm{d}y \\
&= 62(25)\pi \left[200y - \frac{y^2}{2} \right]_0^{200} \\
&= 62(25)\pi \left[200^2 - \frac{200^2}{2} \right] \\
&= 62(25)\pi \left[\frac{200^2}{2} \right] \\
&= 31(25)\pi \cdot 200^2 \text{ lb} \cdot \text{ft} = \boxed{31\,000\,000\pi \text{ lb} \cdot \text{ft}}
\end{aligned}$$

4. (12 points) Let R be the region bounded above by the graph of the function $f(x) = x^3$ and below by the x-axis from $x = 0$ to $x = 3$. Find the centroid of the region.

Solution:

$$M = \int_0^3 x^3 \, dx = \left. \frac{x^4}{4} \right|_0^3 = \frac{81}{4}$$

$$M_x = \frac{1}{2} \int_0^3 (x^3)^2 \, dx = \left. \frac{1}{2} \frac{x^7}{7} \right|_0^3 = \frac{3^7}{14}$$

$$M_y = \int_0^3 x \cdot x^3 \, dx = \left. \frac{x^5}{5} \right|_0^3 = \frac{3^5}{5}$$

$$\overline{x} = \frac{M_y}{M} = \frac{3^5}{5} \cdot \frac{4}{3^4} = \frac{12}{5}$$

$$\overline{y} = \frac{M_x}{M} = \frac{3^7}{14} \cdot \frac{4}{3^4} = \frac{54}{7}$$

The centroid is $\boxed{\left(\frac{12}{5}, \frac{54}{7}\right)}$

5. (a) (6 points) Find the derivative $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tanh^{-1} x)^2$.

Solution: $2\tanh^{-1} x \cdot \dfrac{1}{1-x^2}$

(b) (6 points) Determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

$$a_n = \left(1 - \frac{2}{n}\right)^n$$

Solution: Letting $L = \lim_{n\to\infty} a_n$,

$$\begin{aligned} \ln L = \lim_{n\to\infty} n \ln\left(1 - \frac{2}{n}\right) &= \lim_{n\to\infty} \frac{\ln\left(1-\frac{2}{n}\right)}{n^{-1}} \\ &\stackrel{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1-\frac{2}{n}} \cdot 2n^{-2}}{-n^{-2}} \\ &= -2 \lim_{n\to\infty} \frac{1}{1-\frac{2}{n}} = -2 \end{aligned}$$

Therefore $L = \boxed{e^{-2}}$