1. (8 points) Find the arc length of the curve $y = \frac{1}{3}\left(2x - \frac{1}{3}\right)^{\frac{3}{2}}$ for $\frac{1}{6} \leq x \leq \frac{2}{3}$.

Solution:

$$y' = \frac{1}{2}\left(2x - \frac{1}{3}\right)^{\frac{1}{2}} \cdot 2$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \left(2x - \frac{1}{3}\right)} = \sqrt{2x + \frac{2}{3}}$$

$$L = \int_{\frac{1}{6}}^{\frac{2}{3}} \sqrt{1 + (y')^2}\,\mathrm{d}x = \int_{\frac{1}{6}}^{\frac{2}{3}} \sqrt{2x + \frac{2}{3}}\,\mathrm{d}x$$

$$= \frac{1}{2}\frac{2}{3}\left(2x + \frac{2}{3}\right)^{\frac{3}{2}}\Bigg|_{1/6}^{2/3}$$

$$= \frac{1}{3}\left[2^{3/2} - 1^{3/2}\right] = \boxed{\frac{1}{3}(2\sqrt{2} - 1)}$$

2. (6 points) A force of 20 Newtons will stretch a spring 0.1 meters from its natural length. Find the work (in Newton-meter) required to compress the spring 0.2 meters from its natural length.

Solution: $F = kx \implies 20 = k(0.1) \implies k = 200.$

$$W = \int_0^{0.2} 200x\,\mathrm{d}x = 100x^2\Big|_0^{2/10} = \boxed{4\,\mathrm{N}\cdot\mathrm{m}}$$

3. (10 points) Find the centroid of the region under the curve $y = \cos(x)$, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Solution:

$$M = \int_{-\pi/2}^{\pi/2} \cos x \, \mathrm{d}x = \sin x \Big|_{-\pi/2}^{\pi/2} = 2$$

$$M_x = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos^2 x \, \mathrm{d}x \qquad (\cos 2x = 2\cos^2 x - 1)$$

$$= \frac{1}{4} \int (1 + \cos 2x) \, \mathrm{d}x$$

$$= \frac{1}{4} \left(x + \frac{1}{2} \sin 2x\right) \Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{4}$$

$$\overline{y} = \frac{M_x}{M} = \frac{\pi}{8}$$

Symmetry of the region allows us to conclude that $\overline{x} = 0$. The centroid is $\boxed{(0, \frac{\pi}{8})}$

4. (12 points) Determine whether the sequence converges. If the sequence converges, calculate its limit.

(a) $\{a_n\}_{n=1}^{\infty}$, where $a_n = \left(1 + \frac{2}{n}\right)^{\frac{n}{3}}$

Solution: Let $L = \lim_{n\to\infty} a_n$. Then

$$\begin{aligned}\ln L &= \lim_{n\to\infty} \frac{n}{3} \ln\left(1 + \frac{2}{n}\right) \\ &= \frac{1}{3} \lim_{n\to\infty} \frac{\ln\big(1 + 2/n\big)}{n^{-1}} \\ &\overset{L'H}{=} \frac{1}{3} \lim_{n\to\infty} \frac{\frac{1}{1+2/n} \cdot -2n^{-2}}{-n^{-2}} \\ &= \frac{2}{3} \lim_{n\to\infty} \frac{1}{1 + 2/n} = \frac{2}{3}\end{aligned}$$

Therefore $L = \boxed{e^{2/3}}$

(b) $\{c_n\}_{n=1}^{\infty}$, where $c_n = \frac{1}{n^2} + \sin(n)$

Solution: The sequence $\boxed{\text{does not converge}}$ since $\lim_{n\to\infty} \sin(n)$ does not exist.

5. (6 points) Compute the sum of the series $\sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n-1}}$

Solution:

$$= \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^{n-1} = \frac{\frac{3}{4}}{1 - \frac{3}{4}} = \frac{3}{4} \cdot 4 = \boxed{3}$$

6. (18 points) Determine whether the series converges; list each test of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3+2n+1}{2n^4-n^2+3}$

Solution: Limit Comparison Test with $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{n^3+2n+1}{2n^4-n^2+3} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^4+2n^2+n}{2n^4-n^2+3} = \frac{1}{2}$$

Thus by Limit Comparison Test, the given series also $\boxed{\text{diverges}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \left|\cos\left(\frac{1}{n}\right)\right|$

Solution: $\lim_{n\to\infty} \left|\cos\left(\frac{1}{n}\right)\right| = 1 \neq 0$, so by divergence test, the series $\boxed{\text{diverges}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

Solution: Using ratio test:

$$\begin{aligned}\lim_{n\to\infty} \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} &= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{e^{(n+1)^2-n^2}} \\ &= \lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{1}{e^{2n+1}} \\ &= 1 \cdot 0 = 0 < 1\end{aligned}$$

By ratio test, the series is $\boxed{\text{absolutely convergent}}$