Name Solutions ______________________ Rec. Instr.______________________

Signature ____________________________ Rec. Time______________________

Math 221 – Exam 1 – January 30, 2018

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 75 minutes to complete the exam.

SHOW YOUR WORK!

Problem	Points	Points Possible	Problem	Points	Points Possible
1		20	6		10
2		10	7		10
3		10	8		10
4		10	9		10
5		10	**Total Score**		**100**

$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$

$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$

$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

$$\int \sin^n(x)\,dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

1. Evaluate the following integrals.

A. (5 points) $\int e^{2x}\,dx = \int e^u \cdot \frac{du}{2} = \frac{e^u}{2} + C = \frac{e^{2x}}{2} + C$

$$\boxed{\begin{array}{c} u = 2x \\ du = 2dx \\ \frac{du}{2} = dx \end{array}}$$

B. (5 points) $\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\ln^2(x)}{2} + C$

$$\boxed{\begin{array}{c} u = \ln(x) \\ du = \frac{1}{x}dx \end{array}}$$

C. (10 points) $\int x^2 \ln(x) dx = \ln(x) \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx = \frac{x^3 \ln(x)}{3} - \int \frac{x^2}{3} dx$

$= \frac{x^3 \cdot \ln(x)}{3} - \frac{x^3}{9} + C$

$$\boxed{\begin{array}{cc} \multicolumn{2}{c}{\text{IBP}} \\ u = \ln(x) & dv = x^2 dx \\ du = \frac{1}{x}dx & v = \frac{x^3}{3} \end{array}}$$

2. (10 points) Evaluate $\displaystyle\int_0^1 \frac{12x^2+6}{(2x^3+3x+1)^2}\,dx$.

$$\int_0^1 \frac{12x^2+6}{(2x^3+3x+1)^2}dx = \int_1^6 \frac{2\,du}{u^2} = -\frac{2}{u}\Big|_1^6 = -\frac{2}{6}+\frac{2}{1} = -\frac{1}{3}+2 = \frac{5}{3}$$

$u = 2x^3+3x+1$

$du = (6x^2+3)dx$

x	u
1	6
0	1

3. (10 points) Evaluate $\displaystyle\int x\sqrt{x+5}\,dx$.

$$\int x\sqrt{x+5}\,dx = \int (u-5)\sqrt{u}\,du = \int \left(u^{3/2} - 5\sqrt{u}\right)du$$

$$\boxed{\begin{array}{l} u = x+5 \\ du = dx \\ u-5 = x \end{array}}$$

$$= \frac{2}{5}u^{5/2} - \frac{10}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2} + C$$

4. (10 points) Evaluate $\int e^x \cos(x)\, dx$.

$$I = \int e^x \cos(x)\,dx = \cos(x)e^x - \int e^x(-1)\sin(x)\,dx$$

IBP
$u = \cos(x)$ $\quad dv = e^x dx$
$du = -\sin(x)dx$ $\quad v = e^x$

$$= e^x\cos(x) + \int e^x \sin(x)\,dx$$

$$= e^x\cos(x) + \sin(x)e^x - \int e^x\cos(x)\,dx$$

IBP
$u = \sin(x)$ $\quad dv = e^x dx$
$du = \cos(x)dx$ $\quad v = e^x$

$$= e^x(\cos(x) + \sin(x)) - I$$

$$2I = e^x(\cos(x) + \sin(x)) + C$$

$$I = \frac{e^x(\cos(x) + \sin(x))}{2} + C_1$$

5. (10 points) A particle moving along a straight line has velocity $v(t) = t^2 \cdot e^t$ ft/min after t minutes. How far does the particle travel in the first minute?

By the Net Change Theorem, the distance traveled by the particle in the first minute is:

$$\int_0^1 v(t)\,dt = \int_0^1 t^2 e^t\,dt = t^2e^t\Big|_0^1 - \int_0^1 e^t \cdot 2t\,dt$$

IBP
$u = t^2$ $\quad dv = e^t dt$
$du = 2t\,dt$ $\quad v = e^t$

$$= 1^2e^1 - 0^2e^0 - \int_0^1 2te^t\,dt$$

$$= e - \left((2te^t)\Big|_0^1 - \int_0^1 e^t \cdot 2\,dt\right)$$

IBP
$u = 2t$ $\quad dv = e^t dt$
$du = 2dt$ $\quad v = e^t$

$$= e - \left(2\cdot 1\cdot e^1 - 2\cdot 0\cdot e^0 - (2e^t)\Big|_0^1\right)$$

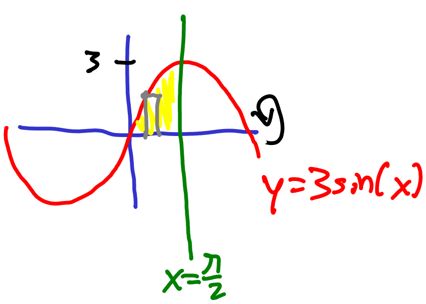
$$= e - (\cancel{2e} - \cancel{2e^1} + 2e^0)$$

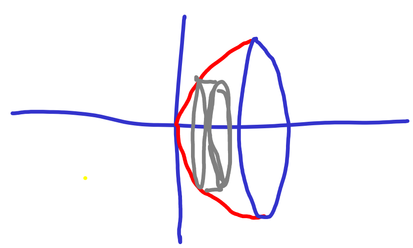
$$= e - 2 \text{ ft}$$

6. (10 points) Evaluate $\int \tan^6(\theta)\sec^4(\theta)\,d\theta$.

$$\int \tan^6(\theta)\sec^4(\theta)\,d\theta = \int \tan^6(\theta)\sec^2(\theta)\sec^2(\theta)\,d\theta$$
$$= \int \tan^6(\theta)(\tan^2(\theta)+1)\sec^2(\theta)\,d\theta$$
$$\boxed{\begin{array}{l} u=\tan(\theta) \\ du=\sec^2(\theta)d\theta \end{array}}$$
$$= \int u^6(u^2+1)\,du$$
$$= \int (u^8+u^6)\,du$$
$$= \frac{u^9}{9}+\frac{u^7}{7}+C$$
$$= \frac{\tan^9(\theta)}{9}+\frac{\tan^7(\theta)}{7}+C$$

7. (10 points) Find the volume of the solid obtained by rotating the region bounded by $y = 3\sin(x)$, the x-axis, and $x = \frac{\pi}{2}$ around the x-axis.





$$\text{Volume} = \int_0^{\frac{\pi}{2}} \pi(3\sin(x))^2\,dx = 9\pi\int_0^{\frac{\pi}{2}}\sin^2(x)\,dx$$

$$\boxed{\sin^2(\theta)=\frac{1-\cos(2\theta)}{2}}$$
$$= 9\pi\int_0^{\frac{\pi}{2}}\frac{1-\cos(2\theta)}{2}\,d\theta$$

(Note, one could instead apply a reduction formula.)

$$\boxed{\begin{array}{l|l} u=2\theta & \begin{array}{c|c} \theta & u \\ \hline \frac{\pi}{2} & \pi \\ 0 & 0 \end{array} \\ du=2d\theta & \\ \frac{du}{2}=d\theta & \end{array}}$$

$$= 9\pi\int_0^{\pi}\frac{1-\cos(u)}{2}\cdot\frac{du}{2} = \frac{9\pi}{4}\int_0^{\pi}(1-\cos(u))\,du$$
$$= \frac{9\pi}{4}(u-\sin(u))\Big|_0^{\pi} = \frac{9\pi}{4}(\pi-\sin(\pi)-0+\sin(0)) = \frac{9\pi^2}{4}$$

8. (10 points) By using a trigonometric substitution, evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2(\theta)\,d\theta}{\sqrt{1+\tan^2(\theta)}} = \int \frac{\sec^2(\theta)\,d\theta}{\sqrt{\sec^2(\theta)}}$$

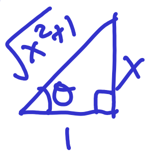
$$\boxed{\begin{array}{l} x = \tan(\theta) \\ dx = \sec^2(\theta)\,d\theta \end{array}}$$

$$= \int \sec(\theta)\,d\theta$$

$$= \ln|\sec(\theta) + \tan(\theta)| + C$$

$$= \ln\left|\sqrt{x^2+1} + x\right| + C$$

$x = \tan(\theta)$



$\sec(\theta) = \sqrt{x^2+1}$

9. (10 points) By using a trigonometric substitution, evaluate $\int \sqrt{9-x^2}\,dx$.

$$\int \sqrt{9-x^2}\,dx = \int \sqrt{9-9\sin^2(\theta)} \cdot 3\cos(\theta)\,d\theta$$

$$\boxed{\begin{array}{l} x = 3\sin(\theta) \\ dx = 3\cos(\theta)\,d\theta \end{array}}$$

$$= \int \sqrt{9(1-\sin^2(\theta))} \cdot 3\cos(\theta)\,d\theta$$

$$= \int 3\sqrt{\cos^2(\theta)} \cdot 3\cos(\theta)\,d\theta$$

$$= 9\int \cos^2(\theta)\,d\theta$$

Reduction Formula

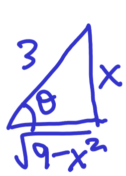
$$= 9\left(\frac{\cos(\theta)\sin(\theta)}{2} + \frac{1}{2}\int d\theta\right)$$

$$= \frac{9}{2}\cos(\theta)\sin(\theta) + \frac{9}{2}\theta + C$$

$\frac{x}{3} = \sin(\theta)$

$\arcsin\left(\frac{x}{3}\right) = \theta$

$\cos(\theta) = \frac{\sqrt{9-x^2}}{3}$



$$= \frac{9}{2} \cdot \frac{\sqrt{9-x^2}}{3} \cdot \frac{x}{3} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C$$

$$= \frac{1}{2} \cdot x \cdot \sqrt{9-x^2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C$$