Name Solutions Rec. Instr.
Signature Rec. Time

Math 221 – Exam 2 – February 27, 2018

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 75 minutes to complete the exam.

SHOW YOUR WORK!

Problem	Points	Points Possible	Problem	Points	Points Possible
1		14	6		10
2		6	7		9
3		6	8		9
4		10	9		9
5		18	10		9

Total Score

$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$

$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$

$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

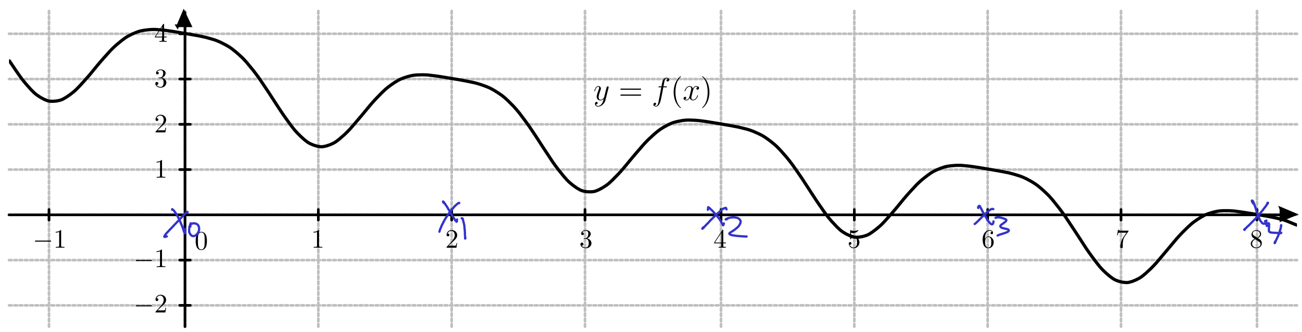
$$\int \sin^n(x)\,dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$M_x = \frac{\rho}{2}\int_a^b \left(f(x)^2 - g(x)^2\right)\,dx \qquad M_y = \rho\int_a^b x\left(f(x)-g(x)\right)\,dx$$



1. (7 points each) $y = f(x)$ is plotted above. Find the following approximations to $\int_0^8 f(x)\,dx$.

$\Delta x = \frac{8-0}{4} = 2$

A. $T_4 = \frac{\Delta x}{2}\left(f(x_0) + 2\left(f(x_1) + f(x_2) + f(x_3)\right) + f(x_4)\right)$

$= \frac{2}{2}\left(4 + 2(3 + 2 + 1) + 0\right)$

$= 16$

B. $S_4 = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$

$= \frac{2}{3}\left(4 + 4\cdot 3 + 2\cdot 2 + 4\cdot 1 + 0\right)$

$= \frac{2}{3}\cdot 24$

$= 16$

2. (6 points) **Write** an integral that calculates the surface area of the volume generated by rotating the curve $y = x^2 + x + 7$ from $x = 0$ to $x = 5$ around the x-axis. **You do not need to evaluate the integral!**

$\frac{dy}{dx} = 2x + 1$

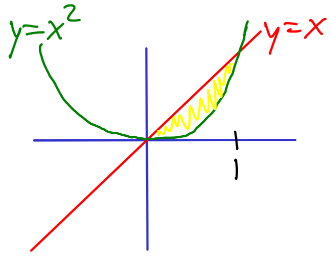
Surface Area $= \int_0^5 2\pi\cdot(x^2 + x + 7)\sqrt{1 + (2x+1)^2}\,dx$

3. (6 points) Find the center of mass $\bar{x}$ if there is a mass of 4 kg at $x = 1$ ft, a mass of 2 kg at $x = 3$ ft, and a mass of 4 kg at $x = 10$ ft.

$$\bar{x} = \frac{4\cdot 1 + 2\cdot 3 + 4\cdot 10}{4+2+4} = \frac{50}{10} = 5 \text{ ft}$$

4. (10 points) Find the center of mass (centroid) for the area between $y = x$ and $y = x^2$.

(You can set the density constant $\rho = 1$ if you want.)



$$\text{Mass} = \rho \int_0^1 (x - x^2)\,dx = \rho\left[\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1\right]$$
$$= \rho\left[\left(\tfrac{1}{2} - \tfrac{1}{3}\right) - (0-0)\right] = \frac{\rho}{6}$$

$$M_x = \frac{\rho}{2}\int_0^1 \left(x^2 - (x^2)^2\right)dx = \frac{\rho}{2}\int_0^1 (x^2 - x^4)\,dx = \frac{\rho}{2}\left[\left(\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^1\right]$$
$$= \frac{\rho}{2}\left[\left(\tfrac{1}{3} - \tfrac{1}{5}\right) - (0-0)\right] = \frac{\rho}{2}\cdot\frac{2}{15} = \frac{\rho}{15}$$

$$M_y = \rho\int_0^1 x(x - x^2)\,dx = \rho\int_0^1 (x^2 - x^3)\,dx = \rho\left[\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^1\right]$$
$$= \rho\left[\left(\tfrac{1}{3} - \tfrac{1}{4}\right) - (0-0)\right] = \frac{\rho}{12}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{\rho}{12}}{\frac{\rho}{6}} = \frac{6}{12} = \frac{1}{2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{\rho}{15}}{\frac{\rho}{6}} = \frac{6}{15} = \frac{2}{5}$$

5. (9 points each) Evaluate the following integrals. (Use proper limit notation.)

A. $\displaystyle\int_0^{1/2} \frac{dx}{x\ln(x)} = \lim_{b\to 0^+}\left[\int_b^{1/2} \frac{dx}{x\ln(x)}\right] = \lim_{b\to 0^+}\left[\int_{\ln(b)}^{\ln(1/2)} \frac{du}{u}\right]$

$u = \ln(x)$, $du = \dfrac{dx}{x}$

x	u
1/2	ln(1/2)
b	ln(b)

$$= \lim_{b\to 0^+}\left[\ln|u|\Big|_{\ln(b)}^{\ln(1/2)}\right] = \lim_{b\to 0^+}\left[\ln\left|\ln\left(\tfrac{1}{2}\right)\right| - \ln|\ln(b)|\right] = -\infty$$

The integral diverges.

B. $\displaystyle\int_0^{\infty} e^{-2x}\,dx = \lim_{b\to\infty}\left[\int_0^b e^{-2x}\,dx\right] = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2x}\Big|_0^b\right]$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2b} + \frac{1}{2}\right] = 0 + \frac{1}{2} = \frac{1}{2}$$

6. (10 points) Find the arc length of $y = \frac{2}{3} \cdot x^{3/2}$ from $x = 0$ to $x = 3$.

$$\frac{dy}{dx} = \sqrt{x}$$

$$\text{Arc Length} = \int_0^3 \sqrt{1 + (\sqrt{x})^2}\,dx = \int_0^3 \sqrt{1+x}\,dx$$

$$= \int_1^4 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}\cdot 1^{3/2}$$

$$= \frac{2}{3}\cdot 8 - \frac{2}{3} = \frac{14}{3}$$

$u = 1+x$

$du = dx$

x	u
3	4
0	1

7. (9 points) Evaluate $\displaystyle\int \frac{3x+5}{x^2+3x+2}\,dx$.

$$\frac{3x+5}{x^2+3x+2} = \frac{3x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$3x+5 = A(x+2) + B(x+1)$

Setting $x=-1$, $2 = A$.

Setting $x=-2$, $-1 = -B$ so $B = 1$.

(One could instead find A and B by comparing coefficients.)

$$\int \frac{3x+5}{x^2+3x+2}\,dx = \int \left(\frac{2}{x+1} + \frac{1}{x+2}\right)dx = 2\ln|x+1| + \ln|x+2| + C$$

8. (9 points) Evaluate $\displaystyle\int \frac{2x^2+2x+1}{x^3-x^2}\,dx$.

$$\frac{2x^2+2x+1}{x^3-x^2} = \frac{2x^2+2x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$2x^2+2x+1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (-A+B)x - B$

$2 = A+C$
$2 = -A+B$ so $B=-1$, $A=-3$, and $C=5$
$1 = -B$

$$\int \frac{2x^2+2x+1}{x^3-x^2}\,dx = \int \left(\frac{-3}{x} - \frac{1}{x^2} + \frac{5}{x-1}\right)dx = -3\ln|x| + \frac{1}{x} + 5\ln|x-1| + C$$

9. (9 points) A spring with a natural length of 1 m requires 10 J to stretch the spring from 1 m to 3 m. How much work is required to stretch the spring from 1 m to 5 m?

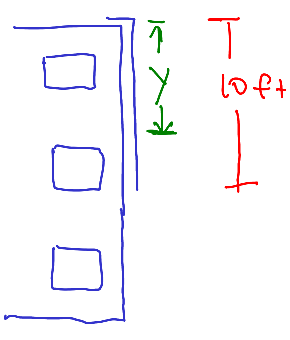
$$10 = \int_0^2 kx\,dx = \frac{kx^2}{2}\Big|_0^2 = 2k - 0 = 2k$$

$$k = 5$$

Work to stretch from 1m to 5 m $= \int_0^4 5x\,dx = \frac{5x^2}{2}\Big|_0^4 = \frac{5\cdot 4^2}{2} - 0$

$$= 40 \text{ J}$$

10. (9 points) Find the work done by winding up a hanging cable of length 10 ft and weight-density 5 lb/ft.



Let y denote the distance to the top of the cable.

$$\text{Work} = \int_0^{10} 5\cdot y\,dy = \frac{5y^2}{2}\Big|_0^{10} = \frac{5\cdot 100}{2} - 0$$

$$= 250 \text{ ft}\cdot\text{lb}$$