Name Solutions Rec. Instr.

Signature Rec. Time

Math 221 – Exam 3 – April 3, 2018

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 75 minutes to complete the exam.

SHOW YOUR WORK!

Problem	Points	Points Possible	Problem	Points	Points Possible
1		14	6		9
2		7	7		9
3		5	8		9
4		12	9		21
5		14	**Total Score**		**100**

$\cosh^2(x) - \sinh^2(x) = 1$

$\frac{d}{dx}\sinh(x) = \cosh(x)$

$\frac{d}{dx}\cosh(x) = \sinh(x)$

$\frac{d}{dx}\text{csch}(x) = -\text{csch}(x)\coth(x)$

$\frac{d}{dx}\text{sech}(x) = -\text{sech}(x)\tanh(x)$

$\frac{d}{dx}\tanh(x) = \text{sech}^2(x)$

$\frac{d}{dx}\coth(x) = -\text{csch}^2(x)$

$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}\text{csch}^{-1}(x) = \frac{-1}{|x|\sqrt{1+x^2}}$

$\frac{d}{dx}\text{sech}^{-1}(x) = \frac{-1}{x\sqrt{1-x^2}}$

$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$

$\frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2}$

1. Evaluate the following:

A. (5 points) $\dfrac{d}{dx}\tanh^{-1}(x^3)\cdot e^x = \dfrac{1}{1-(x^3)^2}\cdot 3x^2\cdot e^x + \tanh^{-1}(x^3)e^x$

$$= \frac{3x^2e^x}{1-x^6} + \tanh^{-1}(x^3)\cdot e^x$$

B. (9 points) $\displaystyle\int \cosh^2(x)\sinh^3(x)\,dx$

$$= \int \cosh^2(x)\sinh^2(x)\sinh(x)\,dx$$

$$= \int \cosh^2(x)(\cosh^2-1)\sinh(x)\,dx$$

$\boxed{\begin{array}{l} u=\cosh(x) \\ du=\sinh(x)dx \end{array}}$

$$= \int u^2(u^2-1)\,du$$

$$= \int (u^4-u^2)\,du$$

$$= \frac{u^5}{5}-\frac{u^3}{3}+C$$

$$= \frac{\cosh^5(x)}{5}-\frac{\cosh^3(x)}{3}+C$$

2. (7 points) Find the smallest value of M that guarantees

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-\sum_{n=1}^{M}\frac{(-1)^n}{n}\right|\le .01.$$

$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-\sum_{n=1}^{M}\frac{(-1)^n}{n}\right|\le\frac{1}{M+1}$. We want $\frac{1}{M+1}\le .01=\frac{1}{100}$

so $100\le M+1$ so $99\le M$. We can take $M=99$.

3. (5 points) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 3^n}{4^n}$.

$$\sum_{n=0}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{3}{4}} = \frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{4}}$$

$$= 2 + 4 = 6$$

4. Consider the differential equation $\displaystyle\frac{dy}{dx} = 2xy$.

A. (3 points) Find the constant solution(s). (These are solutions of the form $y(x) = w$ for some constant w.)

$y(x) = 0$ is a constant solution.

B. (6 points) Find the general solution to the differential equation.

For $y \neq 0$, $\displaystyle\int \frac{dy}{y} = \int 2x\,dx$

$$\ln|y| = x^2 + C$$

$$|y| = e^{\ln|y|} = e^{x^2+C} = e^C \cdot e^{x^2}$$

$$y = (\pm e^C)e^{x^2} = C' \cdot e^{x^2}$$

$y(x) = C' \cdot e^{x^2}$ is the general solution.

(Note, when $C' = 0$, this gives the constant solution $y(x) = 0$.)

C. (3 points) Find the particular solution that satisfies $y(0) = 7$.

$$7 = y(0) = C' \cdot e^{0^2} = C'$$

$$y(x) = 7 \cdot e^{x^2}$$

5. A. (7 points) Use the Squeeze Theorem to evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1}$.

$$\frac{-1}{n^2+1} \leq \frac{(-1)^n}{n^2+1} \leq \frac{1}{n^2+1}$$

$\lim_{n\to\infty} \frac{-1}{n^2+1} = 0$, and $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$.

By the Squeeze Theorem, $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$.

B. (7 points) Evaluate $\lim_{n\to\infty} \frac{n^2}{e^n}$.

$$\lim_{n\to\infty} \frac{n^2}{e^n} \underset{\substack{\text{L'Hopital} \\ \frac{\infty}{\infty}}}{=} \lim_{n\to\infty} \frac{2n}{e^n} \underset{\substack{\text{L'Hopital} \\ \frac{\infty}{\infty}}}{=} \lim_{n\to\infty} \frac{2}{e^n} = 0.$$

6. (9 points) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ diverge, converge absolutely, or converge conditionally? (Justify your answer.)

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by the p-series Test.

$\lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$ and $0 < \frac{1}{(n+1)^{1/2}} < \frac{1}{n^{1/2}}$ for $n \geq 1$.

By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ converges.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ converges conditionally.

7. (9 points) Find an explicit formula for the kth partial sum S_k of $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$, and use it to determine if the infinite series converges or diverges.

$$S_k = \sum_{n=1}^{k} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{k} \left(\ln(n) - \ln(n+1)\right)$$

$$= \left(\ln(1) - \cancel{\ln(2)}\right) + \left(\cancel{\ln(2)} - \cancel{\ln(3)}\right) + \left(\cancel{\ln(3)} - \ln\cancel{(4)}\right) + \cdots + \left(\ln\cancel{(k-1)} - \cancel{\ln(k)}\right) + \left(\cancel{\ln(k)} - \ln(k+1)\right)$$

$$= \ln(1) - \ln(k+1) = -\ln(k+1)$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(-\ln(k+1)\right) = -\infty \quad \text{diverges.}$$

8. (9 points) Use the integral test to determine if $\sum_{n=2}^{\infty} \frac{1}{n\ln^2(n)}$ converges or diverges. (Make sure to use correct limit notation.)

$$\int_2^{\infty} \frac{dx}{x\ln^2(x)} = \lim_{b\to\infty}\left(\int_2^b \frac{dx}{x\ln^2(x)}\right)$$

$$\boxed{\begin{array}{c} u = \ln(x) \\ du = \frac{dx}{x} \\ \begin{array}{c|c} x & u \\ \hline b & \ln(b) \\ 2 & \ln(2) \end{array} \end{array}}$$

$$= \lim_{b\to\infty}\left(\int_{\ln(2)}^{\ln(b)} \frac{du}{u^2}\right)$$

$$= \lim_{b\to\infty}\left(\frac{-1}{u}\Big|_{\ln(2)}^{\ln(b)}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{\ln(b)} + \frac{1}{\ln(2)}\right)$$

$$= \frac{1}{\ln(2)} \quad \text{Converges}$$

By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n\cdot\ln^2(n)}$ Converges.

Note, multiple convergence tests could be applied to these series.

9. (7 points each) Determine if each of the series below converge or diverge. Make sure to write the names of any convergence tests you use and justify why the convergence tests apply.

A. $\displaystyle\sum_{n=1}^{\infty} \frac{5n^3+7n}{4n^3+9}$

$$\lim_{n\to\infty} \frac{5n^3+7n}{4n^3+9} = \frac{5}{4} \neq 0.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{5n^3+7n}{4n^3+9}$ diverges by the Divergence Test.

B. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2+3}$

$$0 \le \frac{6}{n^2+3} \le \frac{6}{n^2} \quad \text{for } n \ge 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2} = 6\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series Test,

By the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2+3}$ converges.

C. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^5+3}{n^6+n}$

$$\lim_{n\to\infty} \frac{\frac{3n^5+3}{n^6+n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^6+3n}{n^6+n} = 3.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series Test.

By the Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{3n^5+3}{n^6+n}$ diverges.

Alternatively, $\displaystyle\sum_{n=1}^{\infty} \frac{3n^5+3}{n^6+n} = \sum_{n=1}^{\infty} \frac{3(n^5+1)}{n(n^5+1)} = \sum_{n=1}^{\infty} \frac{3}{n} = 3\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series Test.