Name Solutions Rec. Instr.
Signature Rec. Time

Math 221 – Final Exam – May 9, 2018

No books, calculators, or notes are allowed. Please make sure that your cell phone is turned off. You will have 110 minutes to complete the exam.

SHOW YOUR WORK!

Problem	Points	Points Possible	Problem	Points	Points Possible
1		25	7		12
2		5	8		5
3		6	9		5
4		6	10		4
5		15	11		6
6		5	12		6

Total Score

$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$

$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$

$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{arcsec}\left(\frac{x}{a}\right) + C$$

$$\int \sin^n(x)\,dx = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\text{Arc Length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{Surface Area} = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$M_x = \frac{\rho}{2}\int_a^b \left(f(x)^2 - g(x)^2\right)\,dx \qquad M_y = \rho\int_a^b x\left(f(x)-g(x)\right)\,dx$$

$$T_N = \frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{N-1})+f(x_N)\right)$$

$$S_N = \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{N-2})+4f(x_{N-1})+f(x_N)\right)$$

$$\text{Work} = \int_a^b F(x)\,dx \qquad \cosh^2(x)-\sinh^2(x)=1 \qquad \frac{d}{dx}\sinh(x)=\cosh(x)$$

$$\frac{d}{dx}\cosh(x)=\sinh(x) \qquad \frac{d}{dx}\text{csch}(x) = -\text{csch}(x)\text{coth}(x) \qquad \frac{d}{dx}\text{sech}(x) = -\text{sech}(x)\tanh(x)$$

$$\frac{d}{dx}\tanh(x)=\text{sech}^2(x) \qquad \frac{d}{dx}\coth(x) = -\text{csch}^2(x) \qquad \frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\cosh^{-1}(x)=\frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}\text{csch}^{-1}(x) = \frac{-1}{|x|\sqrt{1+x^2}} \qquad \frac{d}{dx}\text{sech}^{-1}(x) = \frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tanh^{-1}(x)=\frac{1}{1-x^2} \qquad \frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n} \qquad \arctan(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \qquad (1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n$$

$$\text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad \text{Area} = \pm\int_a^b y(t)x'(t)\,dt$$

$$\text{Arc Length} = \int_\alpha^\beta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

1. (5 points each) Evaluate the following:

A. $\frac{d}{dx}\left(\tanh^{-1}(x^2)\cdot\sinh(x^5)\right) = \frac{1}{1-(x^2)^2}\cdot 2x\cdot\sinh(x^5)+\tanh^{-1}(x^2)\cosh(x^5)\cdot 5x^4$

$$= \frac{2x\cdot\sinh(x^5)}{1-x^4} + 5x^4\tanh^{-1}(x^2)\cosh(x^5)$$

B. $\int x^2\ln(x)\,dx = \ln(x)\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{dx}{x} = \frac{x^3\ln(x)}{3} - \int\frac{x^2}{3}dx$

$$\boxed{\begin{array}{cc} \multicolumn{2}{c}{\text{IBP}} \\ u=\ln(x) & dv=x^2dx \\ du=\frac{dx}{x} & v=\frac{x^3}{3} \end{array}}$$

$$= \frac{x^3\ln(x)}{3} - \frac{x^3}{9} + C$$

C. $\int\frac{1}{x^2+x}dx = \int\frac{1}{x(x+1)}dx$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = A(x+1) + Bx$$

Set $x=0$, $1=A$.

Set $x=-1$, $1=-B$ so $B=-1$.

$$\int\frac{1}{x^2+x}dx = \int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + C$$

D. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 4}{3^n}$ $= \displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + 4\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{2}{3}} + 4\cdot\frac{1}{1-\frac{1}{3}}$

$= \dfrac{1}{\frac{1}{3}} + 4\cdot\dfrac{1}{\frac{2}{3}} = 3 + 4\cdot\frac{3}{2} = 3+6 = 9$

E. $\displaystyle\int \sin^4(\theta)\cos^3(\theta)\,d\theta$ $= \displaystyle\int \sin^4(\theta)\left(1-\sin^2(\theta)\right)\cos(\theta)\,d\theta$

$\boxed{\begin{array}{c} u = \sin(\theta) \\ du = \cos(\theta)\,d\theta \end{array}}$ $\rightarrow$ $= \displaystyle\int u^4(1-u^2)\,du = \int (u^4 - u^6)\,du$

$= \dfrac{u^5}{5} - \dfrac{u^7}{7} + C$

$= \dfrac{\sin^5(\theta)}{5} - \dfrac{\cos^7(\theta)}{7} + C$

2. (5 points) Find the degree-2 Maclaurin polynomial of $f(x) = \ln(1+2x)$.

$f'(x) = \dfrac{2}{1+2x}$ $\quad f''(x) = \dfrac{-4}{(1+2x)^2}$

$f(0) = 0$ $\quad f'(0) = 2$ $\quad f''(0) = -4$

$P_2(x) = 0 + 2\cdot x - \frac{4}{2!}x^2 = 2x - 2x^2$

Alternatively, this can be found by truncating the Maclaurin series $\ln(1+2x) = (2x) - \dfrac{(2x)^2}{2} + \dfrac{(2x)^3}{3} - \dfrac{(2x)^4}{4} + \cdots$

3. (6 points) Find the general solution to $\dfrac{dy}{dx} = (2x+1)y$.

$y(x)=0$ is a constant solution. For $y \neq 0$,

$$\int \frac{dy}{y} = \int (2x+1)\,dx$$

$$\ln|y| = x^2 + x + C$$

$$|y| = e^{x^2+x+C} = e^C e^{x^2+x}$$

$$y = (\pm e^C) e^{x^2+x}$$

The general solution is $y = C' e^{x^2+x}$.

4. (6 points) Use the integral test to determine if $\displaystyle\sum_{n=1}^{\infty} 2ne^{-n^2}$ converges or diverges. (Use proper limit notation.)

$$\int_1^\infty 2xe^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b 2xe^{-x^2}\,dx = \lim_{b\to\infty} \int_{-1}^{-b^2} -e^u\,du$$

$$\boxed{\begin{array}{l|l|l} u = -x^2 & x & u \\ du = -2x\,dx & b & -b^2 \\ -du = 2x\,dx & 1 & -1 \end{array}}$$

$$= \lim_{b\to\infty} \left[-e^u \Big|_{-1}^{-b^2} \right] = \lim_{b\to\infty} \left[-e^{-b^2} + e^{-1} \right] = \frac{1}{e}$$

By the Integral Test, $\sum_{n=1}^{\infty} 2ne^{-n^2}$ converges.

5. (5 points each) Determine if each of the series below converge or diverge. Make sure to write the names of any convergence tests you use and justify why the convergence tests apply.

A. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^5+n}{n^5+7}$

$\displaystyle\lim_{n\to\infty} \frac{3n^5+n}{n^5+7} = 3$ so $\displaystyle\sum_{n=1}^{\infty} \frac{3n^5+n}{n^5+7}$ diverges by the Divergence Test.

B. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

The series alternates, $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, and $0 < \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for $n \geq 1$. By the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Converges.

C. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+8}{5n^3+7}$

$$\lim_{n\to\infty} \frac{\frac{2n+8}{5n^3+7}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^3+8n^2}{5n^3+7} = \frac{2}{5}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges by the p-Series Test.

By the Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2n+8}{5n^3+7}$ converges.

6. (5 points) Determine the interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{5^n(x+3)^n}{n!}$.

$$\lim_{n\to\infty}\left|\frac{\frac{5^{n+1}(x+3)^{n+1}}{(n+1)!}}{\frac{5^n(x+3)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}(x+3)^{n+1}\,n!}{5^n(x+3)^n(n+1)!}\right| = \lim_{n\to\infty}\frac{5|x+3|}{n+1} = 0$$

By the Ratio Test, the power series converges on $(-\infty,\infty)$.

7. (6 points each) Find the Maclaurin series of:

A. $\frac{x}{1-x^2}$ $= x\left(\frac{1}{1-x^2}\right) = x\left(1+x^2+x^4+x^6+\cdots\right)$

$$= x+x^3+x^5+x^7+\cdots$$

$$= \sum_{n=0}^{\infty} x^{2n+1}$$

B. $\int \frac{\sin(x)}{x}\,dx$ $= \int \frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)}{x}\,dx$

$$= \int\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots\right)dx$$

$$= C + x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot(2n+1)!}$$

8. (5 points) Find the arc length of $x(t) = e^t + 3$, $y(t) = 2e^t + 5$ for $0 \le t \le 1$.

$$\text{Arc Length} = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_0^1 \sqrt{(e^t)^2 + (2e^t)^2}\,dt$$

$$= \int_0^1 \sqrt{5e^{2t}}\,dt = \int_0^1 \sqrt{5}\,e^t\,dt = \sqrt{5}\,e^t \Big|_0^1$$

$$= \sqrt{5}\,e - \sqrt{5}$$

9. (5 points) Find the area bounded by the x-axis and the parametric curve $x(t) = \cos(t)$, $y(t) = 2\sin(t)$ for $0 \le t \le \pi$.

For $0 \le t \le \pi$, $y(t) = 2\sin(t) \ge 0$, and $x'(t) = -\sin(t) \le 0$ so $x(t)$ is decreasing.

$$\text{Area} = \int_\pi^0 y(t)x'(t)\,dt = \int_\pi^0 2\sin(t)\cdot(-\sin(t))\,dt = \int_0^\pi 2\sin^2(t)\,dt$$

$$= \int_0^\pi 2\cdot\frac{1-\cos(2t)}{2}\,dt = \int_0^\pi (1-\cos(2t))\,dt$$

$$= \left(t - \frac{\sin(2t)}{2}\right)\Big|_0^\pi = (\pi - 0) - (0 - 0) = \pi$$

10. (4 points) **Write** an integral that calculates the arc length of $r = \theta^2$ from $\theta = 0$ to $\theta = 5$. **You do not need to evaluate the integral!**

$$\text{Arc Length} = \int_0^5 \sqrt{(\theta^2)^2 + (2\theta)^2}\,d\theta = \int_0^5 \sqrt{\theta^4 + 4\theta^2}\,d\theta$$

11. (6 points) Convert $r = 4\sin(\theta)$ to an equation in Cartesian (rectangular) coordinates. State what geometric shape it is, and graph the equation.

$r = 4\sin(\theta)$

$r^2 = 4r\sin(\theta)$

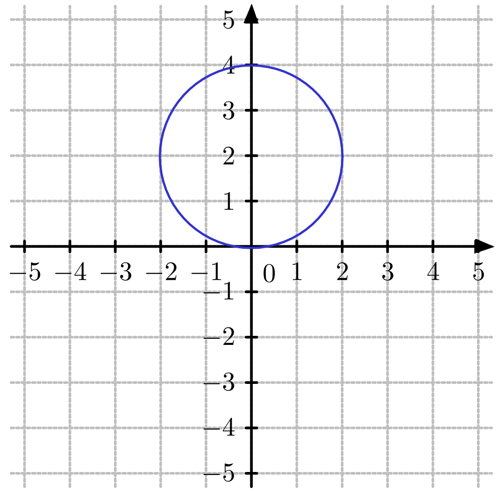
$x^2 + y^2 = 4y$

$x^2 + y^2 - 4y = 0$

$x^2 + y^2 - 4y + 4 = 4$

$x^2 + (y-2)^2 = 4$

This is a circle of radius 2 centered at $(0,2)$



12. (6 points) Below, the polar equation $r = 4\sin(2\theta)$ is graphed. Find the area enclosed by one petal.

Area of a Petal $= \frac{1}{2}\int_0^{\frac{\pi}{2}} (4\sin(2\theta))^2 \, d\theta$

$= \int_0^{\frac{\pi}{2}} 8\sin^2(2\theta) \, d\theta$

$= \int_0^{\frac{\pi}{2}} 8 \cdot \frac{1-\cos(4\theta)}{2} \, d\theta$

$= \int_0^{\frac{\pi}{2}} (4 - 4\cos(4\theta)) \, d\theta$

$= (4\theta - \sin(4\theta)) \Big|_0^{\frac{\pi}{2}}$

$= (2\pi - 0) - (0 - 0)$

$= 2\pi$