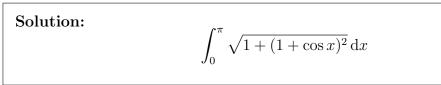
1. (a) (6 points) Write an integral that calculates the length of the curve  $y = x + \sin x$ ,  $0 \le x \le \pi$ . Do not evaluate the integral.



(b) (8 points) Find the surface area of the surface obtained by rotating the curve  $y = \sqrt{x}$ ,  $1 \le x \le 4$  around the *x*-axis. **Evaluate the integral.** 

Solution:		
	$y' = \frac{1}{2\sqrt{x}}$ $(y')^2 = \frac{1}{4x}$	
	$(g) = \frac{1}{4x}$	
Then		
	$SA = \int_{a}^{b} 2\pi f(x) \sqrt{1 + [f'(x)]^2}  \mathrm{d}x$	
	$= \int_{1}^{4} 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} \mathrm{d}x$	
	$=2\pi\int_{1}^{4}\sqrt{x+\frac{1}{4}}\mathrm{d}x$	
	$=2\pi \frac{2}{3} \cdot \left(x+\frac{1}{4}\right)^{3/2} \Big _{1}^{4}$	
	$= \boxed{\frac{4\pi}{3} \left[ \left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2} \right]}$	

2. (14 points) Find the center of mass (centroid)  $(\bar{x}, \bar{y})$  of the region bounded by  $y = e^{-x}$ , x = 0, x = 1 and the x-axis.

Solution:  

$$M = \int_{0}^{1} e^{-x} dx = -e^{-x} \Big|_{0}^{1} = \underbrace{-e^{-1} + 1}_{0}$$

$$M_{x} = \frac{1}{2} \int_{0}^{1} (e^{-x})^{2} dx = \frac{1}{2} \int_{0}^{1} e^{-2x} dx = -\frac{1}{4} \left[ e^{-2x} \right]_{0}^{1} = \underbrace{-\frac{1}{4} (e^{-2} - 1)}_{0}$$

$$M_{y} = \int_{0}^{1} x e^{-x} dx \qquad \begin{bmatrix} D & I \\ + & x & e^{-x} \\ - & 1 & -e^{-x} \\ + & 0 & e^{-x} \end{bmatrix}$$

$$= -xe^{-x} - e^{-x}$$

$$= (-x - 1)^{e^{-x}} \Big|_{0}^{1} = \underbrace{-2e^{-1} + 1}_{0}$$

Thus

$$\bar{x} = \frac{M_y}{M} = \frac{-2e^{-1} + 1}{-e^{-1} + 1} = \boxed{\frac{-2 + e}{-1 + e}}$$
$$\bar{y} = \frac{M_x}{M} = \frac{-\frac{1}{4}(e^{-2} - 1)}{-e^{-1} + 1}$$
$$= -\frac{1}{4}\frac{1 - e^2}{-e^{-1} + e^2} = \frac{1}{4e}\frac{1 - e^2}{1 - e} = \boxed{\frac{1 + e}{4e}}$$

3. (a) (7 points) A spring requires 10J to stretch it 2m from its rest length. How much work is required to stretch the spring from 2m to 4m from its rest length.

Solution:	
	$10 = \int_0^2 kx  dx = k \frac{x^2}{2} \Big _0^2 = 2k \implies k = 5$ $W = \int_2^4 5x  dx = 5 \left[ \frac{x^2}{2} \right]_2^4 = 5(8-2) = \boxed{30J}$

(b) (8 points) Find the work required to pump all the liquid out of a cylindrical tank that has a base of radius 10ft and height 50ft. Use the fact that the density of the liquid is  $\rho$  lb/ft<sup>3</sup>.

## Solution:

$$W = Fd$$
  
=  $V\rho d$   
=  $\rho \pi r^2 dy (50 - y)$   
=  $\int_0^{50} \rho \pi 100(50 - y) dy$   
=  $100 \pi \rho \left[ 50y - \frac{y^2}{2} \right]_0^{50}$   
=  $100 \pi \rho \frac{50^2}{2} = 50^3 \pi \rho = \boxed{125000 \pi \rho \text{ lb ft}}$ 

- 4. Evaluate the following
  - (a) (7 points)  $\frac{\mathrm{d}}{\mathrm{d}x} \tanh^{-1}(\sin(x^2))$ , where  $\tanh^{-1}$  is the inverse function of  $\tanh$ .

$$\frac{1}{1 - (\sin x^2)^2} \cdot \cos\left(x^2\right) \cdot 2x$$

(b) (7 points)  $\int \frac{1}{\sqrt{9+x^2}} dx$ , using the substitution  $x = 3 \sinh \theta$ .

Solution:

Solution:

$$= \int \frac{1}{\sqrt{9(1 + \sinh^2 \theta)}} \cdot 3 \cosh \theta \, d\theta$$
$$= \int \frac{\cosh \theta \, d\theta}{\sqrt{\cosh^2 \theta}} = \int 1 \, d\theta = \theta = \boxed{\sinh^{-1}\left(\frac{x}{3}\right) + C}$$

5. Consider the differential equation

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 y^2.$$

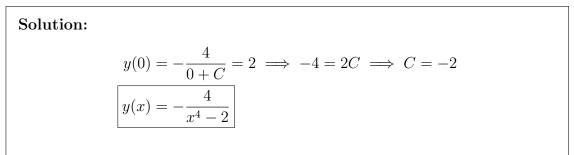
(a) (3 points) Find the constant solutions.

$$0 = x^3 y^2 \implies y = 0$$

(b) (8 points) Find the general solution to the differential equation

Solution:	
	$\int y^{-2}  \mathrm{d}y = \int x^3  \mathrm{d}x$
	$-y^{-1} = \frac{x^4}{4} + C_1 = \frac{x^4 + 4C_1}{4} = \frac{x^4 + C_2}{4}$
	$y(x) = -\frac{4}{x^4 + C}$

(c) (3 points) Find the particular solution satisfying y(0) = 2.



6. (a) (7 points) Evaluate the limit of the sequence  $\lim_{n} \frac{\ln(n)}{n^2}$ .

$$\lim_{n \to \infty} \frac{\ln(n)}{n^2} \stackrel{LH}{=} \lim_{n \to \infty} \frac{\frac{1}{n}}{2n} = \lim_{n \to \infty} \frac{1}{2n^2} = \boxed{0}$$

(b) (8 points) Use the squeeze theorem to calculate  $\lim_{n} \frac{2n - \cos(n)}{n}$ .

and

Solution:

Solution:

$$\lim_{n \to \infty} \left( 2 - \frac{1}{n} \right) = 2$$
$$\lim_{n \to \infty} \left( 2 + \frac{1}{n} \right) = 2$$

Thus by squeeze theorem,

$$\lim_{n \to \infty} \frac{2n - \cos(n)}{n} = \lim_{n \to \infty} \left(2 - \frac{\cos(n)}{n}\right) = \boxed{2}$$

7. Evaluate the series:

$$\begin{array}{l} \text{(a) (7 points)} & \sum_{n=0}^{\infty} \frac{(-1)^n 5 + 2^n}{3^n} \\ \hline \text{Solution:} \\ & \sum_{n=0}^{\infty} \frac{(-1)^n 5 + 2^n}{3^n} = 5 \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\ & = 5 \cdot \frac{1}{1 - \left(-\frac{1}{3}\right)} + \frac{1}{1 - \frac{2}{3}} \\ & = 5 \cdot \frac{1}{4} + \frac{1}{3} \\ & = \frac{15}{4} + 3 = \left[\frac{27}{4}\right] \\ \hline \text{(b) (7 points)} & \sum_{n=3}^{\infty} \frac{1}{n(n-1)} \\ \hline \\ & \text{Solution:} \\ & \frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1} \\ & \text{so we consider} \\ & \sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ & \text{The $k$-th partial sum is} \\ & S_k = \left(\frac{1}{2} - \frac{1}{\beta}\right) + \left(\frac{1}{\beta} - \frac{1}{4}\right) + \dots + \left(\frac{1}{k-2} - \frac{1}{k}\right) + \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ & = \frac{1}{2} - \frac{1}{k} \\ & \text{and the series sums to} \\ & S = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \left(\frac{1}{2} - \frac{1}{k}\right) = \left[\frac{1}{2}\right] \end{array}$$